Lec. 17

Ellipsoids ($\mathcal{E}$)

$$\mathcal{E}(A, \underline{b}, c) := \{\underline{x} \in \mathbb{R}^n \mid \underbrace{x^T A x + 2x^T b + c}_{\text{convex quadratic}} \leq 0\}$$

$A \in \mathbb{S}^n_{++}$

$\underline{b} \in \mathbb{R}^n$

$c \in \mathbb{R}$

$$\boxed{\underline{b}^T A^{-1} \underline{b} - c > 0}$$

$\Updownarrow$

$\mathcal{E}$ is non-empty

$$\mathrm{vol}(\mathcal{E}(A, \underline{b}, c)) = \mathrm{const}(n) \cdot \sqrt{\det\left(\underline{(b^T A^{-1} b - c)}\, A^{-1}\right)}$$

$$(\mathrm{vol}(\cdot))^2 \propto \det(A^{-1})$$

(A, b, c) parameterization of $\mathcal{E}$

$(\underline{q}, Q)$ parameterization of Ellipsoid $\mathcal{E}$

$$\mathcal{E}(\underset{\underset{\mathbb{R}^n}{\Uparrow}}{\underline{q}}, \underset{\underset{\mathbb{S}^n_{++}}{\Uparrow}}{Q}) = \{ \underline{x} \in \mathbb{R}^n \mid (\underline{x} - \underline{q})^T Q^{-1} (\underline{x} - \underline{q}) \leq 1 \}$$

$\underline{q} \equiv$ center vector

$\underline{Q} \equiv$ shape matrix

sq. roots of $\mathrm{eig}(Q)$ = lengths of semi-axes of $\mathcal{E}$

$$\boxed{\mathrm{vol}(\mathcal{E}(\underset{\text{center position}}{\underline{q}}, \underset{\text{shape matrix}}{Q})) = \frac{\mathrm{vol}(B_1^n)}{\sqrt{\det(Q^{-1})}} = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)} \sqrt{\det(Q)}}$$

$$\Leftrightarrow \mathrm{vol}(\mathcal{E}(q, Q)) \propto \sqrt{\det(Q)}$$

$(\sqrt{2})$

$$\frac{\mathrm{vol}(\text{cube})}{\mathrm{vol}(B_1^n)}$$

$$\frac{a^2}{4} = 1 \Rightarrow a = \sqrt{2}$$

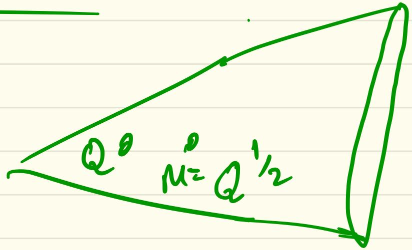
$$\text{Success prob.} = \frac{\mathrm{Vol}(\text{Cube})}{\mathrm{Vol}(B_1^n)} = \frac{(\sqrt{3})^n}{\pi^{n/2}}\,\Gamma\left(\frac{n}{2}+1\right)$$

Yet another parameterization of Ellipsoid

$$\underbrace{\mathcal{E}(\underline{a}, M)}_{\text{Affine image of unit ball}} = \{ \underset{\substack{\uparrow \\ \text{rotation and scaling}}}{M}\underline{v} + \underset{\substack{\uparrow \\ \text{translation}}}{\underline{a}} \mid \underline{v} \in \mathbb{R}^n, \|\underline{v}\|_2 \le 1 \}$$

Relation between $(M \;\&\; Q)$:

$$M = Q^{1/2}$$



Relation between $(A, \underline{b}, c) \longleftrightarrow (\underline{a}, Q)$

$$A = \underbrace{Q^{-1}}_{\succ 0}, \quad \underset{\mathbb{R}^n}{\underline{b}} = -Q^{-1}\underline{a}, \quad c = \underline{a}^T Q^{-1} \underline{a} - 1$$

$\succ 0$

$$\mathcal{E}(A, \underline{b}, c) \longleftrightarrow \mathcal{E}(\underline{a}, Q) \longleftrightarrow \mathcal{E}(\underline{a}, M)$$

$$\longleftrightarrow \boxed{\mathcal{E}(\underline{a}, P) = \{\underline{x} \in \mathbb{R}^n \mid \|P(\underline{x} - \underline{a})\|_2 \le 1\}}$$

Suppose,

$$Q^{-1} = PP, \quad P \succ 0$$

$$\Rightarrow Q = P^{-2}$$

$$\Rightarrow \underbrace{\sqrt{\det(Q)}}\ = \sqrt{(\det(P^{-1}))^2}$$

$$\Rightarrow \mathrm{vol}(\mathcal{E}(a, Q)) \propto \det(P^{-1}) = +(\det(P))^{-1}$$

$$\Rightarrow \log \mathrm{vol}(\mathcal{E}(a, Q)) \propto - \log\det(P)$$

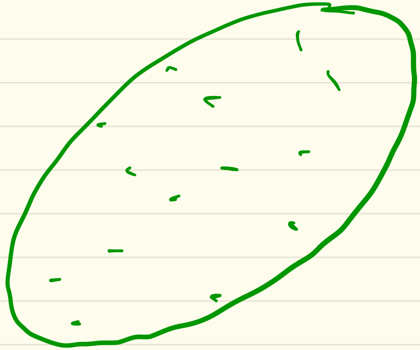
Minimum Vol^{m} Outer Ellipsoid (MVOE)

also known as

Löwner-John Ellipsoid

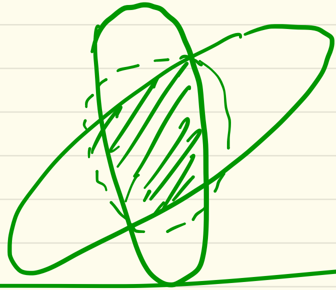
Fritz John

(1948)



For any compact set $\mathcal{S} \subset \mathbb{R}^n$,

the MVOE is unique



$$\mathcal{E}_{LJ} \equiv \mathcal{E}_{MVOE}$$

John's Thm

This inclusion is tight

$$\boxed{\frac{1}{n}\mathcal{E}_{MVOE} \subseteq \mathcal{S} \subseteq \mathcal{E}_{MVOE}}$$

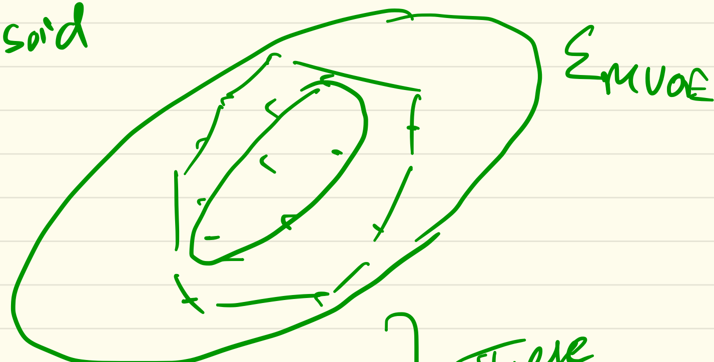
If $\mathcal{S}$ is symmetric ($\Leftrightarrow$ $\underline{x} \in \mathcal{S} \Rightarrow -\underline{x} \in \mathcal{S}$)
then the factor $1/n$ can be improved:

$$\boxed{\frac{1}{\sqrt{n}} \mathcal{E}_{MVOE} \subseteq \mathcal{S} \subseteq \mathcal{E}_{MVOE}}$$

John's Thm.

This inclusion is tight

Max. vol$^{\underline{m}}$ Inner Ellipsoid (MVIE)



$$\text{MVIE} \subseteq \mathcal{S} \subseteq n\,\text{MVIE}$$

for any compact $\mathcal{S} \subset \mathbb{R}^n$

If $\mathcal{S}$ is symmetric, again we have:

$$\text{MVIE} \subseteq \mathcal{S} \subseteq \sqrt{n}\,\text{MVIE}$$

These results are tight

Computing $\mathcal{E}_{MVOE} \equiv \mathcal{E}_{LJ}$

For arbitrary $\mathcal{S}$ (say convex) without loss of generality

(∵ if non-convex set, then take the convex hull)

$$\min_{P \succ 0} \ \log\det(P^{-1})$$

$$\text{s.t.} \quad \mathcal{S} \subseteq \mathcal{E}(\underline{a}, P)$$

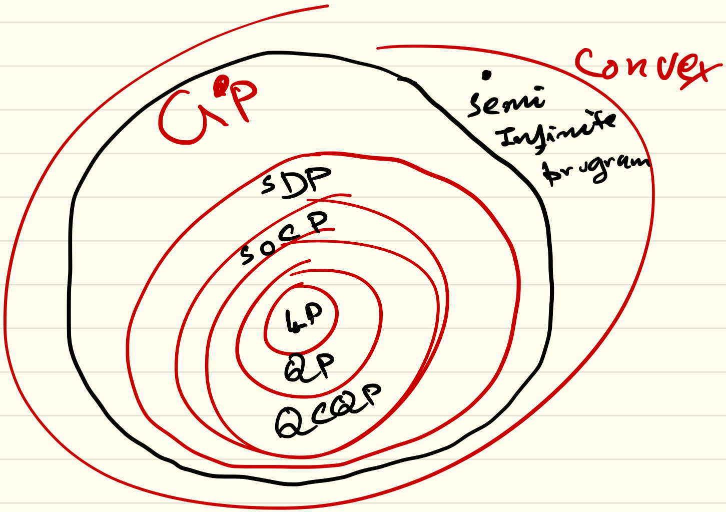
$$\Updownarrow$$

$$\sup_{\underline{x} \in \mathcal{S}} \| P(\underline{x} - \underline{a}) \|_2 \leq 1$$

MVOE problem

semi-infinite programming

$LP \subseteq QP \subseteq QCQP \subseteq SOCP \subseteq SDP \subseteq$ Semi infinite Programs $\subseteq$ Convex



Example: Suppose $\mathcal{S}$ is finite set

$$\mathcal{S} = \{\underset{\underset{\mathbb{R}^n}{\uparrow}}{\underline{a}_1}, \ldots, \underset{\underset{\mathbb{R}^n}{\uparrow}}{\underline{a}_m}\}$$ m points in $\mathbb{R}^n$

Our MVOE Problem becomes:

$$\min_{P \succ 0} \log\det(P^{-1})$$

$$\text{s.t.} \quad \boxed{\| P(\underline{a}_i - \underline{q}) \|_2 \leq 1}$$

$\Updownarrow$ Schur complement

$$\underbrace{\begin{bmatrix} I & P(\underline{a}_i - \underline{q}) \\ (P(\underline{a}_i - \underline{q}))^T & 1 \end{bmatrix}}_{M_i(P, \underline{q})} \succcurlyeq 0, \quad i = 1, \ldots, m$$

$$\leftarrow \begin{bmatrix} \square & [\,] \\ \square & \cdot \end{bmatrix}$$

$$\mathrm{diag}(M_1(P,q), \ldots, M_m(P,q)) \succcurlyeq 0.$$

Interior-point algorithm (can be done in cvx / cvxpy)

Computing MVIE

(Max. Vol^{m} Inner Ellipsoid)

Suppose $\mathcal{C}$ is a convex compact set.

$$\mathcal{E}(\underline{q}, M) := \{ M\underline{v} + \underline{q} \mid \|\underline{v}\|_2 \le 1 \},$$

$$M \succ 0.$$

$$\mathrm{vol}(\mathcal{E}) \propto \det(M)$$

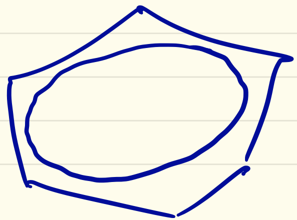
$$\max \mathrm{vol}(\mathcal{E}) \propto \max_{M \succ 0} \det(M) \Leftrightarrow \max_{M \succ 0} \log\det(M)$$

Therefore, MVIE problem:

$$\left.\begin{aligned} &\max_{M \succ 0} \ \log\det(M) \\ &\text{s.t.} \ \underline{\mathcal{E}(a,M) \subseteq \mathcal{S}} \\ &\qquad \Updownarrow \\ &\sup_{\|\underline{v}\|_2 \le 1} \mathbb{1}_{\mathcal{C}}(M\underline{v} + a) \le 0 \end{aligned}\right\}$$ convex problem

Example: Suppose

$$\left.\mathcal{C} = \{\underline{x} \in \mathbb{R}^n \mid a_i^T \underline{x} \le b_i, \ i = 1, \dots, m\}\right\}$$ MVIE inside Polyhedron



$$\boxed{\begin{aligned}&\max \ \log\det(M)\\ &M \succ 0\end{aligned}}$$

$$\mathcal{E}(M,\underline{a}) = \{ M\underline{v} + \underline{a} \mid \|\underline{v}\|_2 \le 1 \}$$

s.t.

$$\mathcal{E} \subseteq \mathcal{C}$$

$$\Updownarrow$$

$$\sup_{\underline{x} \in \mathcal{E}} a_i^T \underline{x} \le b_i \,, \quad i = 1, \ldots, m$$

$$\Updownarrow$$

$$\sup_{\|\underline{v}\|_2 \le 1} a_i^T (M\underline{v} + \underline{a}) \le b_i, \quad i = 1, \ldots, m$$

$$\Updownarrow$$

$$\sup_{\|\underline{v}\|_2 \le 1} a_i^T M \underline{v} + \underbrace{a_i^T \underline{a} \le \underline{b_i}}, \quad i = 1, \ldots, m$$

$$\Updownarrow$$

$$\boxed{\| M \underline{a}_i \|_2 + a_i^T \underline{a} \le b_i, \quad i = 1, \ldots, m}$$

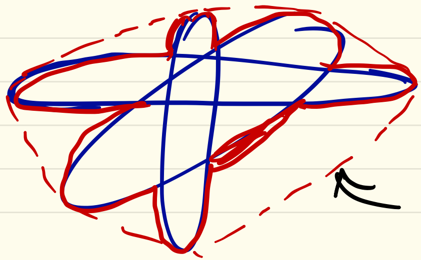
$$\Updownarrow$$

$$\text{s.t.} \begin{bmatrix} (b_i - a_i^T \underline{x}) I & M a_i \\ a_i^T M & (b_i - a_i^T \underline{x}) \end{bmatrix} \succeq 0$$

$i = 1, \dots, m$

can be done in cvx/cvxpy.

union of Ellipsoids (EMVOE can be computed)



Convex hull of the union

Computing $\mathcal{E}_{MVOE}$ for $\bigcup_{i=1}^{m} \mathcal{E}_i$ makes use of the so-called "S-procedure".

Main idea: When is one quadratic inequality a consequence of another (or a set of other) quadratic inequalities?

S-procedure for two quadratic forms:

Suppose $F_0, F_1 \in S^n$.

When is it true that $\forall \underline{z} \in \mathbb{R}^n$, the inequality

$$\underline{z}^T F_1 \underline{z} \geq 0 \Rightarrow \underline{z}^T F_0 \underline{z} \geq 0 \text{ ?}$$

In words,

when does non-negativity of one quadratic form imply non-negativity of other?

Sufficient condition: If $\exists \tau \geq 0$ s.t. $F_0 \geq \tau F_1$

then $\underline{z}^T F_1 \underline{z} \geq 0 \Rightarrow \underline{z}^T F_0 \underline{z} \geq 0$

Proof (Easy): $F_0 \geq \tau F_1 \Rightarrow \underline{z}^T F_0 \underline{z} \geq \underbrace{\underbrace{\tau}_{\geq 0} \underbrace{\underline{z}^T F_1 \underline{z}}_{\geq 0}}_{\geq 0}$. (Proved)

Fact: The converse holds (i.e., the condition is also necessary), provided $\exists \underline{u} \in \mathbb{R}^n$ s.t.

$$\underline{u}^T F_1 \underline{u} > 0$$

Proof: (Not easy)

S-procedure for two quadratic forms with strict inequalities

When does $\underline{z}^T F_1 \underline{z} \geqslant 0,\ \underline{z} \neq 0 \Rightarrow \underline{z}^T F_0 \underline{z} \ngeq 0$?

Answer: Again,

if $\exists\, \tau \geqslant 0$ s.t. $F_0 > \tau F_1$

then the implication holds.

Fact: Again, converse holds provided $\exists\, \underline{u} \in \mathbb{R}^n$ s.t. $\underline{u}^T F_1 \underline{u} > 0$.

slightly general version in Text, Appendix B.2 for quadratic forms $x^T A_i x + 2 b_i^T x + c_i$, etc.
$i = 1, 2$

Also see Example B.1 there (p. 655 in text)

Back to computing $\mathcal{E}_{MVOE} \supseteq \bigcup_{i=1}^{m} \mathcal{E}_i$.

Suppose

$$\mathcal{E}_i = \{\underline{x} \in \mathbb{R}^n \mid \underline{x}^T A_i \underline{x} + 2\underline{b}_i^T \underline{x} + c_i \leq 0\}, \quad i=1,\ldots,m$$

where $A_i \in \mathbb{S}_{++}^n \; \forall\, i=1,\ldots,m$.

For $\mathcal{E}_{MVOE}$, we do the $(\underline{a}, P)$ parameterizn.

Recall: $\mathcal{E}_{MVOE}(\underline{a}, P) = \{\underline{x} \in \mathbb{R}^n \mid \|P(\underline{x}-\underline{a})\|_2 \leq 1\}$

$$\Leftrightarrow \mathcal{E}_{MVOE}(\underline{r}, P) = \{\underline{x} \in \mathbb{R}^n \mid \|P\underline{x} + \underline{r}\|_2 \leq 1\}$$

where $\underline{r} := -P\underline{a} \in \mathbb{R}^n$

So we have $\mathcal{E}_i \equiv \mathcal{E}(A_i, \underline{b}_i, c_i)$, $i=1,\ldots,m$

and $\mathcal{E}_{MVOE} \equiv \mathcal{E}(\underline{r}, P)$

By "S-procedure",

$$\mathcal{E}_i \subseteq \mathcal{E}_{MVOE} \iff \exists\, \tau \geq 0 \text{ s.t.}$$

(if and only if)

$$\begin{bmatrix} (P^2 - \tau A_i) & (P\underline{r} - \tau \underline{b_i}) \\ (P\underline{r} - \tau \underline{b_i})^T & \underline{r}^T\underline{r} - 1 - \tau c_i \end{bmatrix} \preceq 0.$$

Since $\mathrm{vol}(\mathcal{E}_{MVOE}) \propto \det(P^{-1})$, therefore,

$$\text{minimize} \quad \log\det(P^{-1})$$

$$\text{s.t.} \quad \tau_1 \geq 0, \ldots, \tau_m \geq 0$$

$$\begin{bmatrix} (P^2 - \tau_i A_i) & (P\underline{r} - \tau_i \underline{b_i}) \\ (P\underline{r} - \tau_i \underline{b_i})^T & \underline{r}^T\underline{r} - 1 - \tau_i c_i \end{bmatrix} \preceq 0, \quad i = 1, \ldots, m.$$

Introducing new variables $\underline{\tilde{r}} := P\underline{r}$, $\tilde{P} := P^2$,

$$\text{minimize} \quad \log\det(A^{-1})$$

$$\text{s.t.} \quad \tau_1 \geq 0, \ldots, \tau_m \geq 0$$

$$\underbrace{\left.\begin{matrix} m \\ \text{LMIs} \\ \text{in} \\ \left(\tilde{P}, \underline{\tilde{r}}, \{\tau_i\}_{i=1}^{m}\right) \end{matrix}\right\}}\begin{bmatrix} (\tilde{P} - \tau_i A_i) & (\underline{\tilde{r}} - \tau_i b_i) & 0 \\ (\underline{\tilde{r}} - \tau_i b_i) & (-1 - \tau_i c_i) & \underline{\tilde{r}}^T \\ 0 & \underline{\tilde{r}} & -\tilde{P} \end{bmatrix} \leq 0, \quad i = 1, \ldots, m$$

which is a convex problem in variables:

$\tilde{P}$, $\underline{\tilde{r}}$, $\tau_1, \ldots, \tau_m$

Affine invariance of $\mathcal{E}_{LJ}$ (both $\mathcal{E}_{MVOE}$ and $\mathcal{E}_{MVIE}$)

Given compact $\mathcal{S} \subset \mathbb{R}^n$.

Consider $T \in \mathbb{R}^{n \times n}$ nonsingular. Then

$$\mathcal{E}_{LJ}(T\mathcal{S}) = T\,\mathcal{E}_{LJ}(\mathcal{S})$$