Non-Convex problems : Lecture # 2

$$\min \ \text{rank}(X)$$

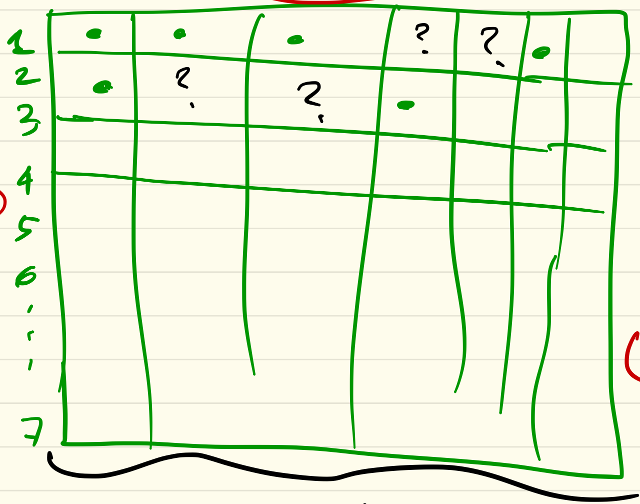
subject to $X_{ij} = M_{ij}$

$\forall (i,j) \in \Omega$ set of already watched movies

(given)

users

Movies



Netflix Problem

(Matrix completion problem)

X

$m \times n$

$$\min \ \|X\|_*$$

subject to $X_{ij} = M_{ij}$

sum of singular values of X

called "Nuclear norm" of matrix X

Functions of scalars, vectors & matrices

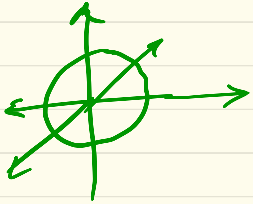
$$f : \mathcal{X} \mapsto \mathbb{R}$$

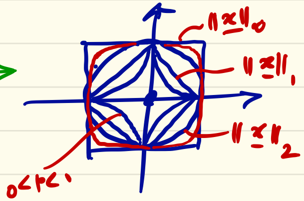
$$f(x) \longleftrightarrow \underset{\text{(scalar)}}{x} \in \mathcal{X} \subseteq \mathbb{R}$$

$$f(\underline{x}) \longleftrightarrow \underset{\text{(vector)}}{\underline{x}} \in \mathcal{X} \subseteq \mathbb{R}^n$$

$$f(X) \longleftrightarrow \underset{\text{(matrix)}}{X} \in \mathcal{X} \subseteq \mathbb{R}^{m \times n}$$

Geometrically, vector 2-norm



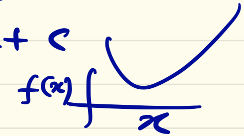


Plots for

$$\|\underline{x}\|_p \leq 1$$

Examples of $\underbrace{\text{Scalar } f^{\text{ns}} \text{ of scalars}}_{f(x)}$

① $f(x) = ax^2 + bx + c$



② $f(x) = \sin(x)$

③ $f(x) = \log(x)$

Examples of $\underbrace{\text{scalar } f^{\text{ns}} \text{ of vectors}}_{f(\underline{x})}$

① $f(\underline{x}) = \underline{a}^T \underline{x} = \langle \underline{a}, \underline{x} \rangle$

② $f(\underline{x}) = \underline{x}^T \underline{x} = \|\underline{x}\|_2^2$

2-norm

$= x_1^2 + x_2^2 + \dots + x_n^2$

③ $f(\underline{x}) = \|\underline{x}\|_1 = \sum_{i=1}^{n} |x_i|$

④ $f(\underline{x}) = \|\underline{x}\|_\infty = \max_{i=1,\dots,n} |x_i|$

⑤ $f(\underline{x}) = \|\underline{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$

p-norm for any $p > 0$

Scalar functions of matrices X : $f(X)$

Examples: $X \in \mathcal{X} \subseteq \mathbb{R}^{n\times n}$ (square real matrices)

① $f(X) = \operatorname{tr}(X)$

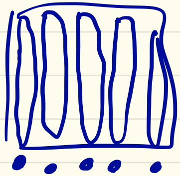
② $f(X) = \det(X)$

③ spectral radius of X $= \rho(X) := \max_{i=1,\dots,n} |\lambda_i(X)|$

Examples: $X \in \mathcal{X} \in \mathbb{R}^{m\times n}$ (rectangular real matrices)

④ $f(X) = \|X\|_1 = \max_{j=1,\dots,n} \sum_{i=1}^{m} |x_{ij}|$

matrix 1-norm (max col. sum)



⑤ $f(X) = \|X\|_\infty$

matrix ∞-norm (max. row sum)

$= \max_{i=1,\dots,n} \sum_{j=1}^{n} |x_{ij}|$

⑥ $f(X) = \|X\|_F$

$= \operatorname{tr}(XX^T)$

$= \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}^2$

called "Frobenius norm" of matrix X

Sometimes, we will need matrix f^{n}s of matrices

$$F : \mathcal{X} \subseteq \mathbb{R}^{n\times n} \longmapsto \mathcal{Y} \subseteq \mathbb{R}^{n\times n}$$

$$\boxed{X \in \mathcal{X}, \quad Y \in \mathcal{Y}, \quad Y = F(X)}$$

Examples : ① $Y = F(X) = X^T$

② $Y = F(X) = X^{-1}$ (assuming $\det(X) \neq 0$)

③ $Y = F(X) = X^p$, $p \in \{0, 1, 2, \ldots\}$

④ $Y = F(X) = \underbrace{\exp(X)}_{\text{matrix exponential}} = \sum_{k=0}^{\infty} \frac{1}{k!} X^k$

Special square matrices :

Identity matrix : I

Diagonal " : $D = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix}$

Symmetric " : $X^T = X$

Skew-Symmetric " : $X^T = -X$

Orthogonal " : $\boxed{X^T X = I = X X^T} \Longleftrightarrow \boxed{X^{-1} = X^T}$

$\det(X^T X) = \det(I)$

$\Rightarrow (\det(X))^2 = 1$

$\Rightarrow \boxed{\det(X) = \pm 1}$

3 different ways to define an Orthogonal matrix