Lecture # 3

Positive (semi)definite matrices

If $X = X^T \in \mathbb{R}^{n\times n}$, $\quad \underline{x}^T X \underline{x} \geq 0 \quad \forall\, \underline{x} \in \mathbb{R}^n$

then we say X is positive (semi)-definite matrix

$X, Y \xrightarrow[\text{partial order}]{\text{(Löwner)}}$ $\quad X \,(\succeq)\, Y \iff X - Y \succeq 0$

$(\preceq) \iff X - Y \preceq 0$

$X = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \qquad \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \qquad (x_1 \;\; x_2)\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$= (x_1 \;\; x_2)\begin{pmatrix} x_1 + x_2 \\ -x_1 + x_2 \end{pmatrix}$

$= x_1^2 + \cancel{x_1 x_2} - \cancel{x_1 x_2} + x_2^2$

$= x_1^2 + x_2^2 \geq 0$

Affine & Convex Sets :

$\underline{x}_1, \underline{x}_2 \in \mathbb{R}^n$

$\underline{x}_1 \neq \underline{x}_2$

Affine sets :

Straight line connecting $\underline{x}_1$ & $\underline{x}_2$ $:= \{ \underline{y} \in \mathbb{R}^n \mid \underline{y} = \theta \underline{x}_1 + (1-\theta)\underline{x}_2, \ \theta \in \mathbb{R} \}$

$\theta = 0 \Longleftrightarrow \underline{y} = \underline{x}_2$

$\theta = 1 \Longleftrightarrow \underline{y} = \underline{x}_1$

(closed straight line segment) $\{ \underline{y} \in \mathbb{R}^n \mid \underline{y} = \theta \underline{x}_1 + (1-\theta)\underline{x}_2, \ 0 \leq \theta \leq 1 \}$

Affine Set : A set $\mathcal{C} \subseteq \mathbb{R}^n$ is affine if $\forall \underline{x}_1, \underline{x}_2 \in \mathcal{C}$ & $\theta \in \mathbb{R}$, $\boxed{\theta \underline{x}_1 + (1-\theta)\underline{x}_2} \in \mathcal{C}$

We say, $\mathcal{C}$ contains $\boxed{\text{linear combination}}$ of any two points if the weights $\boxed{\text{sum as } 1}$.

We can extend linear combⁿ of k points : $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_k$

$$\left\{ \begin{array}{l} \theta_1 \underline{x}_1 + \theta_2 \underline{x}_2 + \ldots + \theta_k \underline{x}_k \\ \text{s.t.} \quad \boxed{\theta_1 + \theta_2 + \ldots + \theta_k = 1} \end{array} \right.$$

affine combination of k points.

If $\mathcal{C}$ is affine & $\underline{x}_0 \in \mathcal{C}$, then consider the set

$$\mathcal{V} := \mathcal{C} - \underline{x}_0 := \{ \underline{x} - \underline{x}_0 \mid \underline{x} \in \mathcal{C} \}$$

a "subspace"

The set $\mathcal{V}$ is closed under "sum" & "scalar multiplication"

(Prove that if $\underline{v}_1, \underline{v}_2 \in \mathcal{V}$, then $\alpha \underline{v}_1 + \beta \underline{v}_2 \in \mathcal{V}$ $\forall \alpha, \beta \in \mathbb{R}$)

Affine set can then alternatively be defined as:

$$\mathcal{C} = \underbrace{\mathcal{V} + \underline{x}_0}_{\text{subspace plus an offset}} = \{ \underline{v} + \underline{x}_0 \mid \forall \underline{v} \in \mathcal{V} \}$$

Example: Sol^n set of linear eq^n $\underbrace{A}_{m\times n}\underbrace{\underline{x}}_{n\times 1} = \underbrace{\underline{b}}_{m\times 1}$

(i.e.) $\mathcal{C} = \{\underline{x} \in \mathbb{R}^n \mid A\underline{x} = \underline{b}\}$

$A\underline{x}_1 = b$

$A\underline{x}_2 = b$

$A(\theta \underline{x}_1 + (1-\theta)x$

$\cdots$

$\mathcal{V}$ = nullspace of matrix A
or kernel

$= \{\underline{x} \in \mathbb{R}^n \mid A\underset{n\times 1}{\underline{x}} = \underline{0}_{m\times 1}\}$

Affine hull of $\mathcal{C}$: ($\mathcal{C}$ itself may not be an affine set)

$\mathrm{aff}(\mathcal{C})$

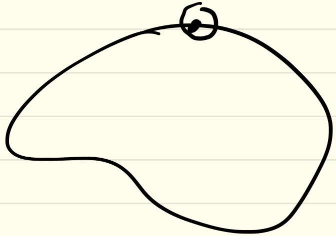
$:= \{\theta_1 \underline{x}_1 + \cdots + \theta_k \underline{x}_k \mid \underline{x}_1, \ldots, \underline{x}_k \in \mathcal{C},\ \sum_{i=1}^{k} \theta_i = 1\}$

= smallest affine set containing $\mathcal{C}$

Affine dimension of $\mathcal{C}$ = Dim. of $\mathrm{aff}(\mathcal{C})$

$$\mathrm{relint}(\mathcal{C}) := \{ \underline{x} \in \mathcal{C} \mid B(\underline{x}, r) \cap \mathrm{aff}(\mathcal{C}) \subseteq \mathcal{C} \text{ for some } r > 0 \}$$

relative interior of $\mathcal{C}$



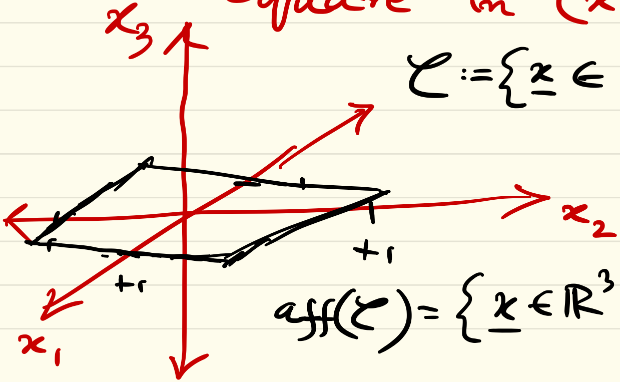
$$B(\underline{x}, r) := \{ \underline{y} \in \mathbb{R}^n \mid \| \underline{y} - \underline{x} \|_2 \leq r \}$$

Relative boundary of $\mathcal{C}$ = $\mathrm{cl}(\mathcal{C}) \setminus \mathrm{relint}(\mathcal{C})$

closure of $\mathcal{C}$

Example:

Square in (x_1, x_2) plane in $\mathbb{R}^3$



$$\mathcal{C} := \{ \underline{x} \in \mathbb{R}^3 \mid -1 \leq x_1 \leq 1,\ -1 \leq x_2 \leq 1,\ x_3 = 0 \}$$

$$\mathrm{aff}(\mathcal{C}) = \{ \underline{x} \in \mathbb{R}^3 \mid x_3 = 0 \}$$

$$\mathrm{relint}(\mathcal{C}) = \{ \underline{x} \in \mathbb{R}^3 \mid -1 < x_1 < +1,\ -1 < x_2 < +1,\ x_3 = 0 \}$$

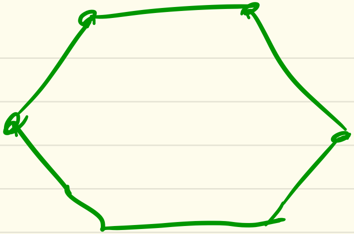
relative bndy. = wire-frame outline

$$= \{ \underline{x} \in \mathbb{R}^3 \mid \max \{|x_1|, |x_2|\} = 1, \; x_3 = 0 \}$$

<u>Convex set</u> : If line segment connecting any two points from $\mathcal{C}$, is also within $\mathcal{C}$, then $\mathcal{C}$ is convex set.

If $\boxed{\forall \underline{x}_1, \underline{x}_2 \in \mathcal{C}}$, & $\theta \underline{x}_1 + (1-\theta)\underline{x}_2 \in \mathcal{C}$,

$$0 \le \theta \le 1.$$



Convex

All affine sets are convex

Convex combination:

$$\sum_{i=1}^{k} \theta_i \underline{x}_i \in \mathcal{C} \quad \text{s.t.} \quad \sum_{i=1}^{k} \theta_i = 1, \quad \theta_i \geq 0$$

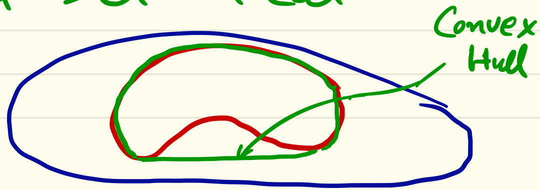
$p(x)$, $p : \mathbb{R}^n \mapsto \mathbb{R}$, $p \geq 0$, $\int p\,dx = 1$, $\int \underline{x}\, p(x)\, dx \in \mathcal{C}$

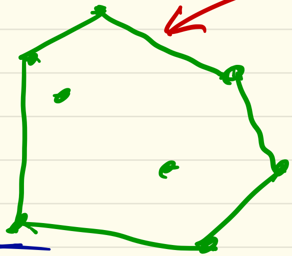
$\mathbb{E}[x]$

θ

Convex Hull:

$$\underbrace{\mathrm{conv}(\mathcal{C})}_{\text{smallest convex set that contains } \mathcal{C}} = \left\{ \theta_1 \underline{x}_1 + \dots + \theta_k \underline{x}_k \;\middle|\; \underline{x}_i \in \mathcal{C},\ \theta_i \geq 0,\ \sum_{i=1}^{k} \theta_i = 1 \right\}$$

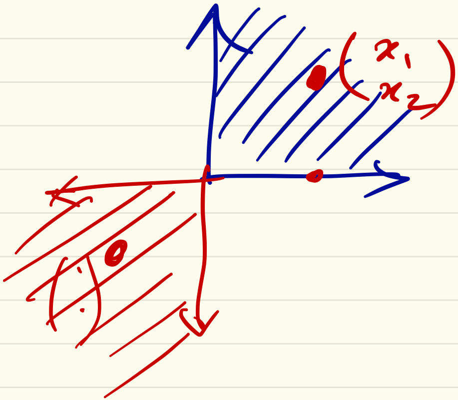




convex Hull of finite set of points

Cones: $\mathcal{C} \subseteq \mathbb{R}^n$ is a cone if

$$\underline{x} \in \mathcal{C},\ \forall\, \theta \geq 0,\ \theta \underline{x} \in \mathcal{C}.$$



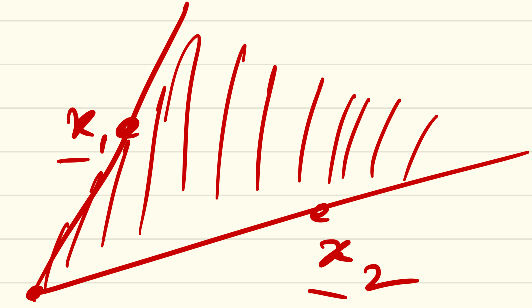
$\mathbb{R}^n_{\geq 0}$ (non-neg. orthant)

$$\underbrace{\mathbb{R}^2_{\geq 0}}_{1^{st}} \cup \underbrace{\mathbb{R}^2_{\leq 0}}_{3^{rd}\text{ quad}}$$

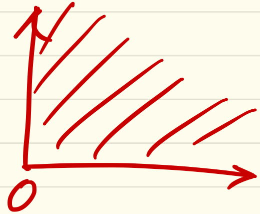
Convex cone : Cone that is also convex.

(i.e.) $\forall\ \underline{x}_1, \underline{x}_2 \in \mathcal{C}$, $\theta_1, \theta_2 \geq 0$,

$$\theta_1 \underline{x}_1 + \theta_2 \underline{x}_2 \in \mathcal{C}$$



2D pie slice

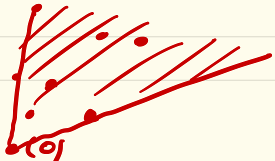


Conic combination : (non-neg. lin. combination)

$$\sum_{i=1}^{k} \theta_i x_i, \quad \theta_i \geq 0.$$

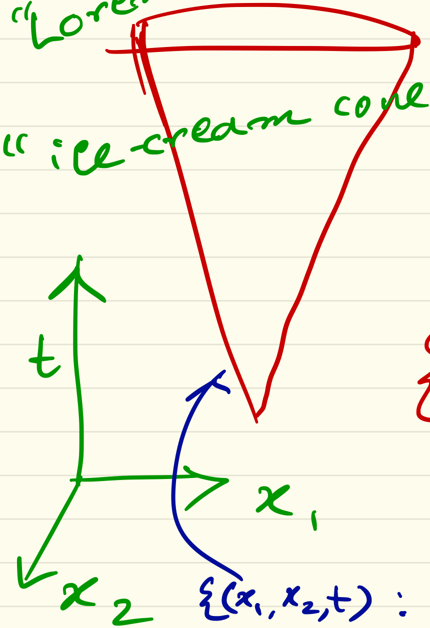
Conic Hull

(smallest convex cone containing $\mathcal{C}$)



Norm balls & Norm cones

"2nd order cone"
"Lorentz cone"
"ice-cream cone"



$$\{x \in \mathbb{R}^n \mid \|x - x_c\| \le r\}$$

any norm

$$\{(x, t) \in \mathbb{R}^{n+1} \mid \|x\| \le t\} \subseteq \mathbb{R}^{n+1}$$

some norm

$$\{(x_1, x_2, t) : \sqrt{x_1^2 + x_2^2} \le t,\ t \ge 0\}$$

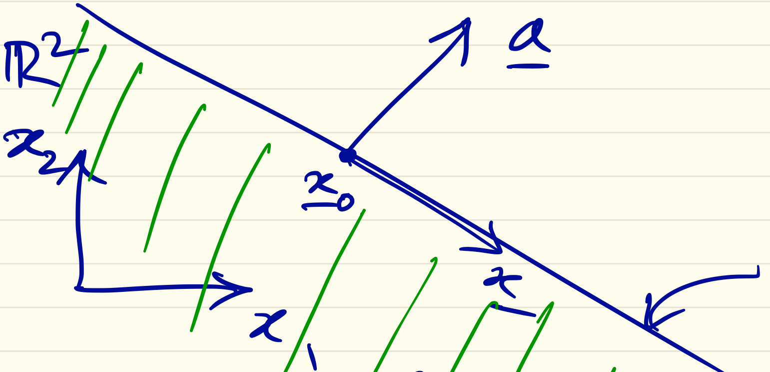
If this is 2-norm

$$= \left\{ \begin{pmatrix} x_{2\times 1} \\ t \end{pmatrix} : \begin{pmatrix} x \\ t \end{pmatrix}^T \begin{bmatrix} I_{2\times 2} & 0_{2\times 2} \\ 0 & -1 \end{bmatrix} \begin{pmatrix} x \\ t \end{pmatrix} \le 0,\ t \ge 0 \right\}$$

Hyperplane & Halfspace :

$$\{ \underline{x} \in \mathbb{R}^n \mid \underline{a}^T \underline{x} = b, \ \underline{a} \in \mathbb{R}^n \setminus \{\underline{0}\}, \ b \in \mathbb{R} \}$$

$$\{ \underline{x} \in \mathbb{R}^n \mid \underline{a}^T \underline{x} \leq b, \ \underline{a} \in \mathbb{R}^n \setminus \{\underline{0}\}, \ b \in \mathbb{R} \}$$



$\underline{a}^T \underline{x} = b$

$(\underline{x} - \underline{x}_0) \perp^r$ to $\underline{a}$