Lecture #4

Euclidean balls & Ellipsoids:

$$\underbrace{B(\underline{x}_c, r)}_{\text{convex}} := \{\underline{x} \in \mathbb{R}^n \mid \|\underline{x} - \underline{x}_c\|_2 \le r\}$$

$$= \{\underline{x} \in \mathbb{R}^n \mid (\underline{x} - \underline{x}_c)^T(\underline{x} - \underline{x}_c) \le r^2\}$$

$$= \{\underline{x}_c + r\underline{u} \mid \underbrace{\underline{u} \in \mathbb{R}^n, \|\underline{u}\|_2 \le 1}\}$$

$\overbrace{\text{Ellipsoid:}}^{\text{convex}}$

$$\mathcal{E}(\underline{x}_c, P) := \{\underline{x} \in \mathbb{R}^n \mid (\underline{x} - \underline{x}_c)^T P^{-1} (\underline{x} - \underline{x}_c) \le 1\}$$

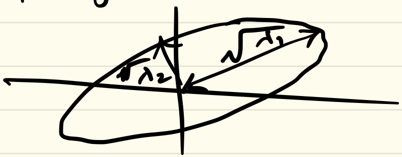
$\underline{x}_c \in \mathbb{R}^n$, $P \in S^n_{++}$

$$= \{\underline{x}_c + M\underline{u} \mid \underline{u} \in \mathbb{R}^n, \|\underline{u}\|_2 \le 1\}$$

$M = P^{1/2}$

$\lambda_i = \text{eig}(P)$

semi-axes length $= \sqrt{\lambda_i}$



$MM = P$

$P = r^2 I \to$ Ball $B(\underline{x}_c, r)$

Polytopes / Polyhedra (Sol^{n}: set of finite number of linear equations & linear inequalities)

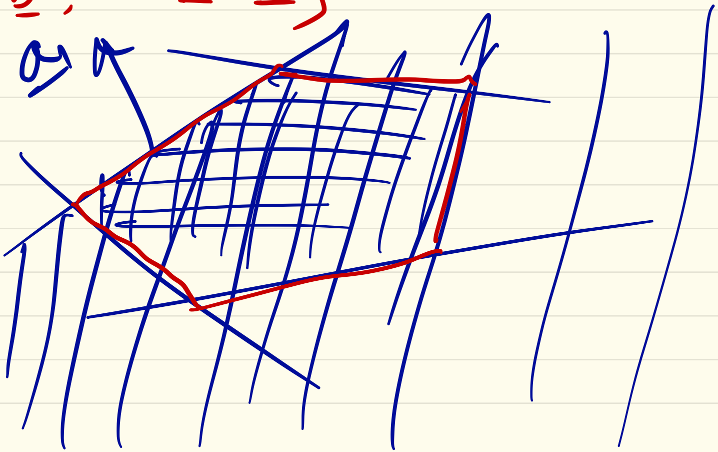
$$\mathcal{P} := \left\{ \underline{x} \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \underline{a}_j^T \underline{x} \le b_j, \quad j = 1, \ldots, m \\ \underline{c}_j^T \underline{x} = d_j, \quad j = 1, \ldots, p \end{array} \right\}$$

= Finite intersections of halfspaces & hyperplanes

$$= \{ \underline{x} \in \mathbb{R}^n \mid A\underline{x} \le \underline{b},\ C\underline{x} = \underline{d} \}$$

$$A_{m \times n} = \begin{pmatrix} \underline{a}_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{pmatrix}$$

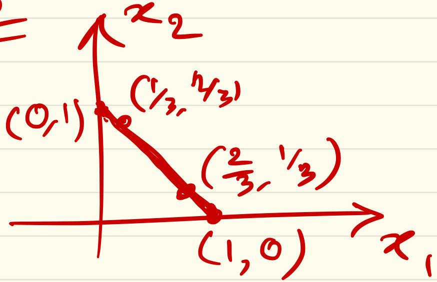
$$C_{p \times n} = \begin{pmatrix} \underline{c}_1^T \\ \vdots \\ c_p^T \end{pmatrix}$$



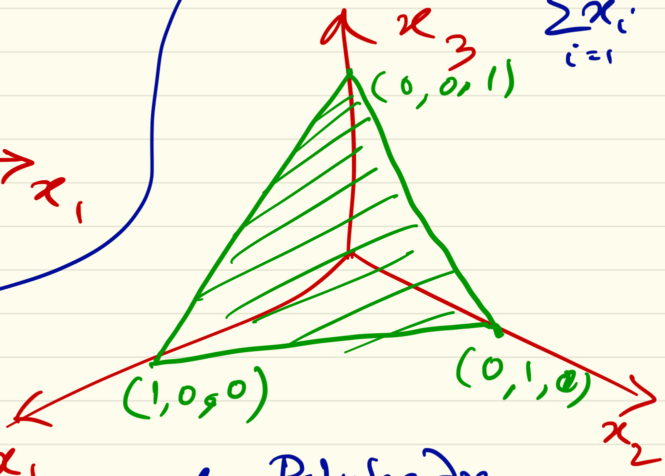
Example of Polyhedron : Standard simplex (Probability simplex)

$$\mathcal{P} = \{ \underline{x} \in \mathbb{R}^n \mid \underline{x} \geq 0 , \; \underbrace{\mathbb{1}^T \underline{x} = 1}_{\sum_{i=1}^{n} x_i} \}$$

If $n = 2$



If $n = 3$



$$\mathbb{1} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}_{n\times 1}$$

Alternative Description of Polyhedron:

$$\mathcal{P} = \{ \theta_1 \underline{v}_1 + \theta_2 \underline{v}_2 + \dots + \theta_k \underline{v}_k \mid \theta_1 + \dots + \theta_m = 1, \; \theta_i \geq 0, \; i = 1, \dots, k, \; m \leq k \}$$

Alternative Description of Polyhedron:

$$P = \{ \theta_1 \underline{v}_1 + \theta_2 \underline{v}_2 + \dots + \theta_k \underline{v}_k \mid \theta_1 + \dots + \theta_m = 1, \ \theta_i \geq 0, \ i = 1, \dots, k, \ m \leq k \}$$

= Non-neg. linear comb^n of $\underline{v}_i$ (vectors) but only the 1st m coefficients sum to 1.

= Convex hull of pt.s $\underline{v}_1, \dots, \underline{v}_m$

& conic " " " $\underline{v}_{m+1}, \dots, \underline{v}_k$.

l_∞ unit norm ball:

$$\mathcal{C} = \{\underline{x} \in \mathbb{R}^n \mid |x_i| \leq 1, \; i = 1, \dots, n\}$$

Intersection of 2n linear inequalities

$$\pm \underline{e}_i^T \underline{x} \leq 1$$

Alternatively

$$\mathcal{C} = \mathrm{conv}\{\underline{v}_1, \dots, \underline{v}_{2^n}\}.$$

Operations preserving set convexity:

① <u>Intersection</u> (Countable OR uncountable)

e.g. $\mathcal{S} = \{\underline{x} \in \mathbb{R}^m \mid \left|\sum_{k=1}^{m} x_k \cos(kt)\right| \leq 1 \text{ for all } -\pi/3 \leq t \leq \pi/3\}$

$$\mathcal{S} = \bigcap_{|t| \leq \pi/3} \mathcal{S}_t$$

where

$$S_t := \left\{ \underline{x} \in \mathbb{R}^n \;\middle|\; -1 \le x_1 \cos(t) + x_2 \cos(2t) + \dots + \cos(nt) \le +1 \right\}$$

$$S = \bigcap_{-\pi/3 \le t \le \pi/3} S_t$$

② Affine Transformation : $A\underline{x} + \underline{b}$

e.g. Projection is convex
Scaling " "
Translation " "
Rotation " "

any combination of these operations is also convex

③ Cartesian Product of convex is convex

$$\underbrace{S_1}_{\text{convex}} \times \underbrace{S_2}_{\text{convex}} = \{ (\underline{x}, \underline{y}) \mid \underline{x} \in S_1, \underline{y} \in S_2 \}$$

Appl^n: Set Sum (Minkowski Sum)

$$\underbrace{S_1}_{\text{convex}} + \underbrace{S_2}_{\text{convex}} = \{ \underline{x} + \underline{y} \mid \underline{x} \in S_1, \underline{y} \in S_2 \}$$

$S_1 + S_2 =$ Image of $S_1 \times S_2$ under function $f(\underline{x}, \underline{y}) = \underline{x} + \underline{y}$

$$= \left[I \;\middle|\; I \right] \begin{pmatrix} \underline{x} \\ \underline{y} \end{pmatrix}$$

Perspective Function :

$$p : \underbrace{\mathrm{dom}(p)}_{\substack{\shortparallel \\ \mathbb{R}^n \times \mathbb{R}_{++} \\ (\mathbb{R}_{>0})}} \longmapsto \mathbb{R}^n$$

$$\mathbb{R}^n \times \mathbb{R}_{++} \subset \mathbb{R}^{n+1}$$

$$p(\underset{\mathbb{R}^n}{\underline{z}}, \underset{\mathbb{R}_{>0}}{t}) = \underline{z}/t$$

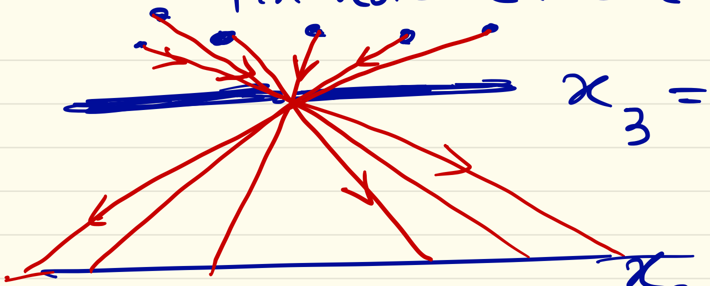
$$\underline{z} = \begin{pmatrix} 1 \\ -2.7 \\ 3 \end{pmatrix}$$

$$t = 4$$

$$p\begin{pmatrix} \underline{z} \\ t \end{pmatrix} = p\left(\begin{pmatrix} 1 \\ -2.7 \\ 3 \\ 4 \end{pmatrix}\right)$$

$$= \begin{pmatrix} 1/4 \\ -2.7/4 \\ 3/4 \end{pmatrix}$$

Pin-hole Camera



$x_3 = 0$

$x_3 = -1$

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longrightarrow \begin{pmatrix} -x_1/x_3 \\ -x_2/x_3 \\ \cancel{-1} \end{pmatrix}$$

chopping off the last

$P(\mathcal{C})$ is convex if $\mathcal{C}$ is convex

Linear fractional function (g)

= Perspective f^n $\circ$ Affine function

$$g : \mathbb{R}^n \mapsto \mathbb{R}^{m+1} \;\Big|\; g(\underline{x}) = \begin{pmatrix} A_{m \times n} \\ \underline{c}^T \end{pmatrix} \underline{x} + \begin{pmatrix} \underline{b} \\ d \end{pmatrix}, \quad b \in \mathbb{R}^m,\ \underline{c} \in \mathbb{R}^n,\ d \in \mathbb{R}$$

$$f(x) = \frac{A\underline{x} + \underline{b}}{\underline{c}^T \underline{x} + d}$$

If $\underline{c} = \underline{0}$ and $d > 0$ then $\mathrm{dom}(f) = \mathbb{R}^n$

$$\mathrm{dom}(f) = \{ \underline{x} \mid \underline{c}^T \underline{x} + d > 0 \}$$

Linear fractional f^n. preserves convexity.
Affine is spl. case of linear fractional.

e.g. conditional probability.

u discrete random var. on $\{1, \ldots, n\}$
v " " " on $\{1, \ldots, m\}$

$$p_{ij} = \text{Joint Probability} = \mathbb{P}(u = i, v = j)$$

Conditional probability $(f_{ij}) = \mathbb{P}(u = i \mid v = j)$

$$= \frac{p_{ij}}{\sum_{i=1}^{n} p_{ij}}$$

Linear fractional function

Cone K is called "proper" if it is

1. convex
2. closed
3. solid (interior is non-empty)
4. pointed (contains no line i.e., $\underline{x} \in K$, & $-\underline{x} \in K$ $\Downarrow$ $\underline{x} = \underline{0}$)

Any proper cone has "partial" ordering.

$$\underline{x} \succeq_K \underline{y} \iff \underline{x} - \underline{y} \in K$$

$$\underline{x} \preceq_K \underline{y} \iff \underline{y} - \underline{x} \in K$$

$$\underline{x} \succ \underline{y} \iff \underline{x} - \underline{y} \in \operatorname{int}(K)$$

e.g. 1) $K = \mathbb{R}^n_+ \Leftrightarrow \succcurlyeq$ (componentwise vector inequality)

$\underline{x}, \underline{y} \in \mathbb{R}^n_+, \quad \underline{x} \succcurlyeq \underline{y}$

$(\mathbb{R}^n_{\geq 0})$

2) $K := \left\{ \underline{c} \in \mathbb{R}^n \;\middle|\; \underline{c}^T \begin{pmatrix} 1 \\ t \\ \vdots \\ t^{n-1} \end{pmatrix} = c_1 + c_2 t + c_3 t^2 + \dots + c_n t^{n-1} \geq 0 \quad \forall\, t \in [0,1] \right\}$

Prove that K is a proper cone.

What does $\succcurlyeq$ mean?

$\underline{c}, \underline{d} \in K$

$\underline{c} \succcurlyeq_K \underline{d} \Leftrightarrow c_1 + c_2 t + \dots + c_n t^{n-1} \geq d_1 + d_2 t + \dots + d_n t^{n-1} \quad \forall\, t \in [0,1]$