Lecture #5

How to visualize S_+^n for $n = 2$

(set of all 2×2 symmetric pos. semi-definite matrices)

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad$ $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$, $\quad$ trace $= a + c \geq 0$

$= \lambda_1 + \lambda_2 \geq 0$

$x^T_{2\times 1} \begin{bmatrix} a & b \\ b & c \end{bmatrix} x_{2\times 1} \geq 0$

$ac \geq b^2$

$\iff \underbrace{ac - b^2}_{\text{determinant}} \geq 0$

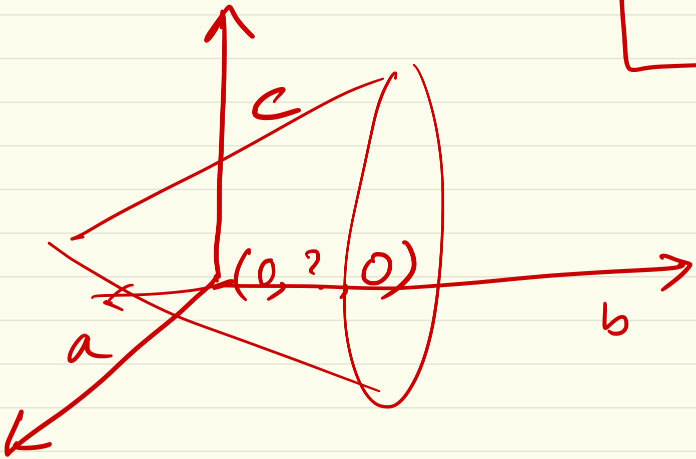
$= \lambda_1 \lambda_2$

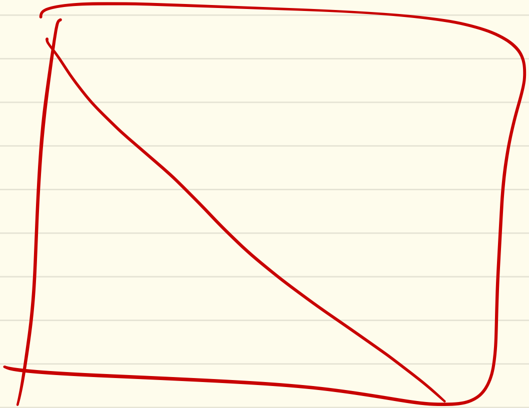
$(x \quad y) \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \geq 0$

$ax^2 + 2bxy + cy^2 \geq 0$

$a \geq 0$ $\quad \longrightarrow \quad$ $c \geq 0$

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} \succeq 0 \iff \boxed{\begin{array}{l} a \geq 0 \\ c \geq 0 \\ ac - b^2 \geq 0 \end{array}}$$





Cone inequalities have nice properties:

$\succcurlyeq_K$ (\succeq) and $\preccurlyeq_K$ (\preceq) preserved under

① Addition ($\underline{x}, \underline{y}, \underline{u}, \underline{v} \in K$)

If $\underline{x} \succcurlyeq \underline{y}$ and $\underline{u} \succcurlyeq \underline{v}$

then $x + u \succcurlyeq y + v$

② Nonneg. scaling ($x, y \in K$)

If $x \succcurlyeq y$ then $\alpha x \succcurlyeq \alpha y \ \forall \alpha \geqslant 0$.

③ transitive: If $x \succcurlyeq y$ and $y \succcurlyeq z$

then $x \succcurlyeq z$

④ reflexive $x \succcurlyeq x$, ⑤ antisymmetric

If $x \succcurlyeq y$ and $y \succcurlyeq x$ then $x = y$

Dual Cone

Suppose K is a cone

(For now, assume $K \subseteq \mathbb{R}^n$)

Polar Cone

$$K^* := \{ \underline{y} \in \mathbb{R}^n \mid \langle \underline{y}, \underline{x} \rangle \geq 0 \ \forall \underline{x} \in K \}$$

K^* is convex even if K is NOT.

$$K^\circ = -K^*$$

(= negative of dual cone)

$$K^\circ = \{ \underline{y} \in \mathbb{R}^n \mid \langle \underline{y}, \underline{x} \rangle \leq 0 \ \forall \underline{x} \in K \}$$

e.g. $K = \mathbb{R}^n_{\geq 0}$ (self dual)

$$K^* = \mathbb{R}^n_{\geq 0}$$

$$K^\circ = \mathbb{R}^n_{\leq 0}$$

Another example: $K = S^n_+$ (self dual), $K^* = S^n_+$

$$\langle X, Y \rangle = tr(X^T Y), \quad X \in K = S^n_+$$
$$= tr(XY)$$

Claim:

$tr(XY) \geq 0 \ \forall X \succcurlyeq 0 \iff Y \succcurlyeq 0$

Proof: (Example 2.24 in book)

Dual of a norm cone:

$$K = \{(\underline{x}, t) \in \mathbb{R}^{n+1} \mid \|\underline{x}\|_p \leq t\}$$

$$K^* = \{(\underline{x}, t) \in \mathbb{R}^{n+1} \mid \|\underline{x}\|_q \leq t\}, \text{ where } 1/p + 1/q = 1$$

- $K_1 \subseteq K_2 \Longrightarrow K_2^* \subseteq K_1^*$
- $K^{**} = (K^*)^* = cl(conv(K))$

(If K was a convex closed cone then $K^{**} = K$)

Separating Hyperplane Thm.

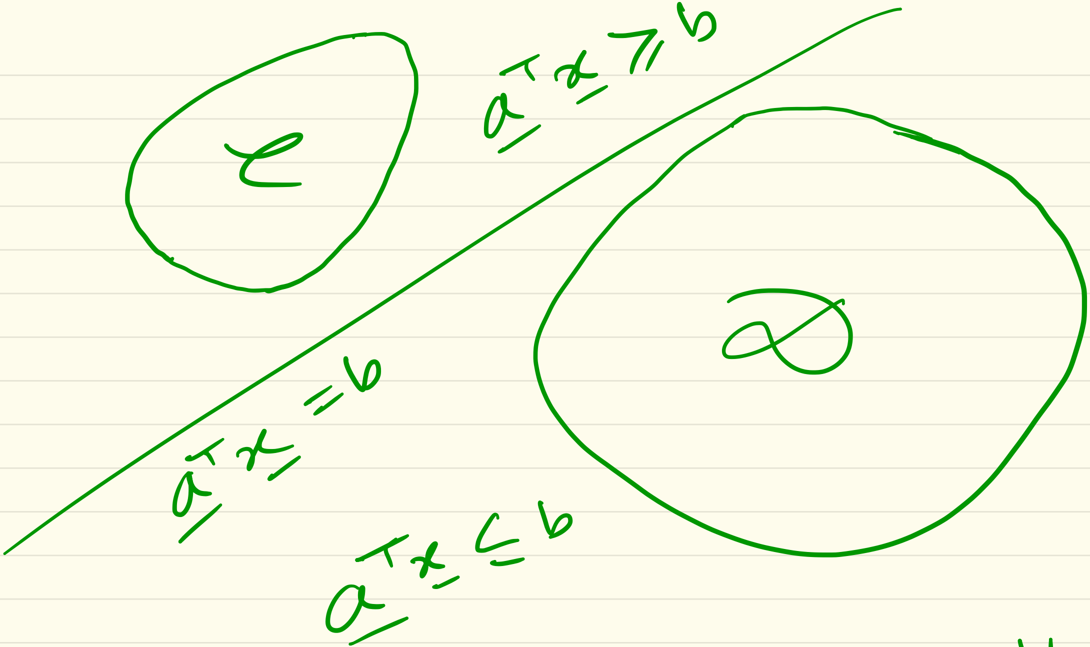
Statement: Let $\mathcal{C}, \mathcal{D} \subset \mathbb{R}^n$ s.t. both $\mathcal{C}$ & $\mathcal{D}$ are convex, and $\mathcal{C} \cap \mathcal{D} = \phi$.

Then $\exists$ $\underline{a} \neq \underline{0} \in \mathbb{R}^n$ & $b \in \mathbb{R}$, s.t.

$$\underline{a}^T \underline{x} \leq b \quad \forall \ \underline{x} \in \mathcal{C}$$

and $$\underline{a}^T \underline{x} \geq b \quad \forall \ \underline{x} \in \mathcal{D}$$

$\underline{x} \in \mathbb{R}^n$



(Proof: book)

We say $\mathcal{C}$ & $\mathcal{D}$ are <u>separable</u> by the hyperplane $\underline{a}^T \underline{x} = b$

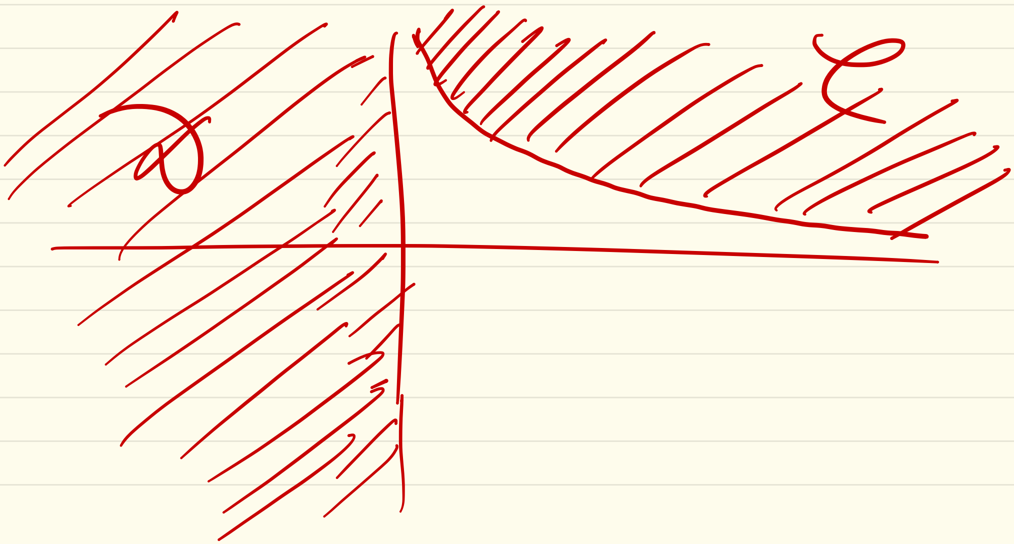
We say <u>strictly separable</u> if $\underline{a}^T \underline{x} > b \ \forall \underline{x} \in \mathcal{C}$

and $\underline{a}^T \underline{x} < b \ \forall \underline{x} \in \mathcal{D}$

Example of Disjoint Convex Sets NOT strictly separable:

$$\mathcal{C} := \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^n \;\middle|\; x_1 x_2 \geq 1, \; x_1, x_2 > 0 \right\}$$

$$\mathcal{D} := \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \;\middle|\; x_1 \leq 0 \right\}$$



Converse is NOT true in general

(i.e. Existence of $\underline{a}^T \underline{x} = b$ s.t. $\underline{a}^T \underline{x} \geq b$ $\forall \underline{x} \in \mathscr{C}$

& $\underline{a}^T \underline{x} \leq b$ $\forall \underline{x} \in \mathscr{D}$

$\not\Rightarrow C \cap \mathscr{D} = \emptyset$

unless you add extra condition on one of the sets

$\rightsquigarrow$ If at least one of the sets $\mathscr{C}$ or $\mathscr{D}$ is also open, then CONVERSE is true . .

Supporting Hyperplane:

Suppose $\mathcal{C} \subseteq \mathbb{R}^n$ and $\underline{x}_0 \in \underset{\sim}{bd}(\mathcal{C})$
boundary
$(bd(\mathcal{C}) := cl(\mathcal{C}) \setminus int(\mathcal{C}))$

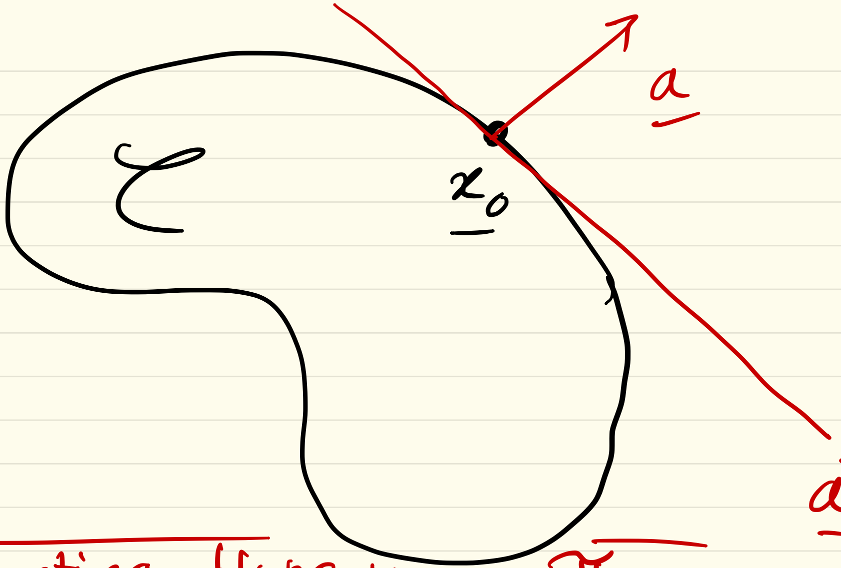
If $\underline{a} \neq \underline{0} \in \mathbb{R}^n$ satisfies

$$\underline{a}^T \underline{x} \leq \underline{a}^T \underline{x}_0 \quad \forall\ \underline{x} \in \mathcal{C}$$

then the hyperplane $\{\underline{x} \in \mathbb{R}^n \mid \underline{a}^T \underline{x} = \underline{a}^T \underline{x}_0\}$ is called a <u>supporting hyperplane</u> to $\mathcal{C}$ @ $\underline{x}_0$.

$\Leftrightarrow$ The point $\underline{x}_0 \in \mathbb{R}^n$ & the set $\mathcal{C}$ are separated by the hyperplane

$\Leftrightarrow$ The hyperplane is tangent to $\mathcal{C}$ @ $\underline{x}_0$ and the halfspace $\{\underline{x} \mid \underline{a}^T \underline{x} \leq \underline{a}^T \underline{x}_0\}$ contains $\mathcal{C}$.

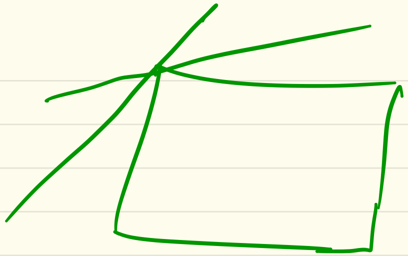


$\underline{a}^T \underline{x} = \underline{a}^T \underline{x}_0$

Supporting Hyperplane Thm.

For any non-empty convex set $\mathcal{C} \subset \mathbb{R}^n$ and any $\underline{x}_0 \in bd(\mathcal{C})$,

$\exists$ a supporting hyperplane to $\mathcal{C}$ @ $\underline{x}_0$



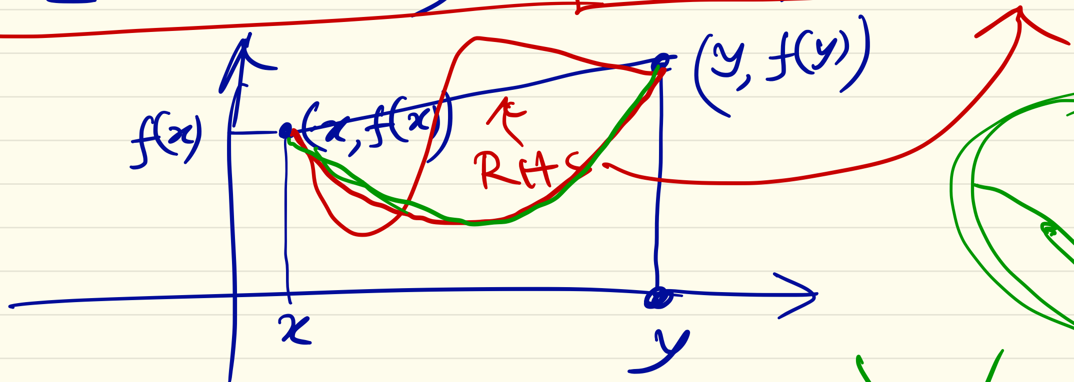
(Partial converse): If a set $\mathcal{C}$ is
→ closed
→ has non-empty interior
→ has supporting hyperplane @ every $\underline{x}_0 \in bd(\mathcal{C})$

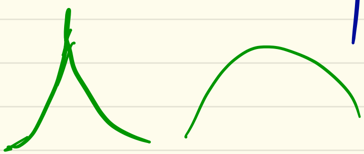
Then $\mathcal{C}$ is convex.

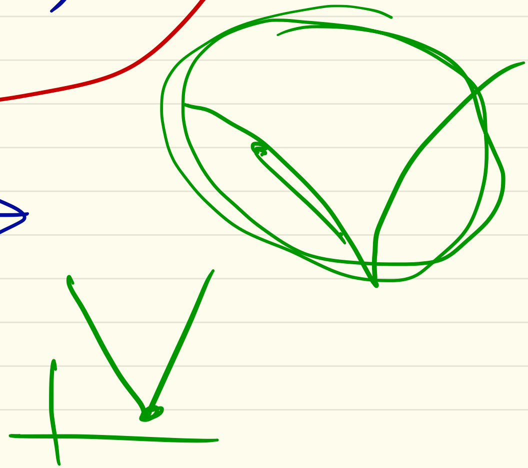
Convex Functions

A f^{n}. $f : \mathbb{R}^n \mapsto \mathbb{R}$ is convex if dom(f) is a convex set, and $\forall\ \underline{x}, \underline{y} \in \text{dom}(f)$ and $0 \leq \theta \leq 1$, we have

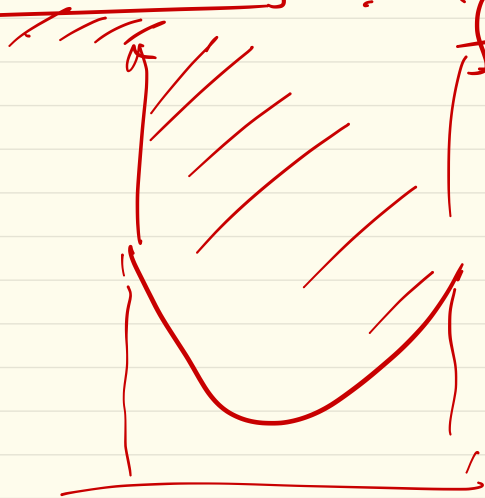
$$\boxed{f(\theta \underline{x} + (1-\theta)\underline{y}) \leq \boxed{\theta f(\underline{x}) + (1-\theta) f(\underline{y})}}$$







Extended defn :



$$\tilde{f}(\underline{x}) = \begin{cases} f(\underline{x}) & \forall\ \underline{x} \in \mathrm{dom}(f) \\ \infty & \forall\ x \notin \mathrm{dom}(f) \end{cases}$$

$$\min_{x \in \mathcal{C}} f(\underline{x})$$

$$= \min_{x \in \mathbb{R}^n} f(x) + \tilde{\mathbb{1}}_{\mathcal{C}}$$

e.g. $f(x) = \mathbb{1}_{\mathcal{C}} := 0 \quad \forall\ \underline{x} \in \mathcal{C}$

where $\mathcal{C}$ is convex set

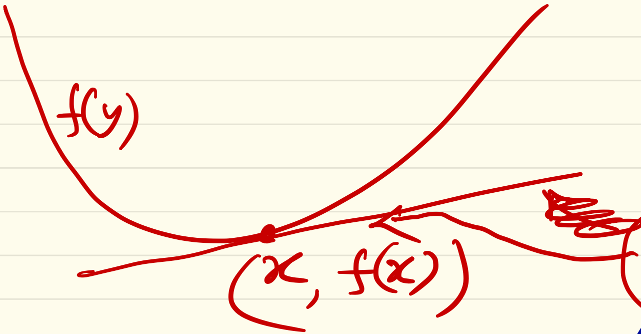
$$\tilde{\mathbb{1}}_{\mathcal{C}}(\underline{x}) = \begin{cases} 0 & \forall\ \underline{x} \in \mathcal{C} \\ \infty & \forall\ \underline{x} \notin \mathcal{C} \end{cases}$$

If f is differentiable (∇f exists $\forall\ \underline{x} \in dom(f)$)

then f is convex if and only if

$$f(\underline{y}) \geq f(\underline{x}) + \left(\nabla f(\underline{x})\right)^T (\underline{y} - \underline{x})$$

$$\forall\ \underline{x}, \underline{y} \in dom(f)$$



$f(\underline{x}) + \left(\nabla f(\underline{x})\right)^T (\underline{y} - \underline{x})$

1st order Taylor approximation of f near x

If convex then 1^{st} order Taylor approxm^{n} is global underestimator.

Converse is also true.

from Local $\rightarrow$ we can establish global property (convexity)