Lecture #6

2^{nd} order condition:

Assumption: f is twice differentiable

(i.e.) $\nabla^2 f$ (Hessian of f) exists @ each pt. in dom(f)

Then f is convex (concave) $\iff$ (iff) ① dom(f) is convex

② $\nabla^2 f \succeq 0 \ \forall \underline{x} \in \underline{\text{dom}(f)}$

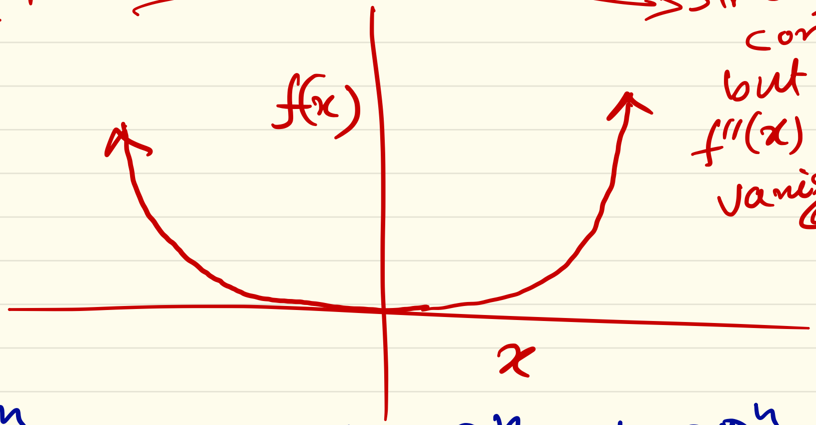
(If $f: \mathbb{R} \mapsto \mathbb{R}$, then $\frac{d^2 f}{dx^2} \geq 0 \ \forall x \in \text{dom}(f)$)

If $\nabla^2 f \succ 0$ ($\Leftrightarrow \nabla^2 f \in S^n_{++}$) $\forall \underline{x} \in \text{dom}(f)$, then f is strictly convex

Converse fails.

Counterexample : $f : \mathbb{R} \mapsto \mathbb{R}$

$f(x) = x^4$ → Strictly convex but $f''(x) = 12x^2$ vanishes @ $x = 0$.



Example :

① $f : \mathbb{R}^n \mapsto \mathbb{R}$, $\quad A \in \mathbb{S}^n$, $\underline{b} \in \mathbb{R}^n$, $c \in \mathbb{R}$

$$f(\underline{x}) = \frac{1}{2} \underline{x}^T A \underline{x} + \underline{b}^T \underline{x} + c$$

$$\nabla^2 f(\underline{x}) = A \succeq 0$$

f is convex $\iff A \succeq 0$

concave $\iff A \preceq 0$

strictly convex $\iff A \succ 0$

concave $\iff A \prec 0$

$$f(\underline{x}) = \frac{1}{2} \underline{x}^T A \underline{x}$$

$$\nabla f = \frac{1}{2} \nabla (\underline{x}^T A \underline{x})$$

$$= \frac{1}{2} (A + A^T) \underline{x}$$

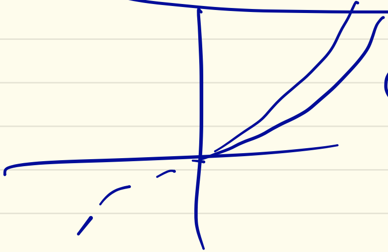
$$= A \underline{x}$$

$$\nabla^2 f = A$$

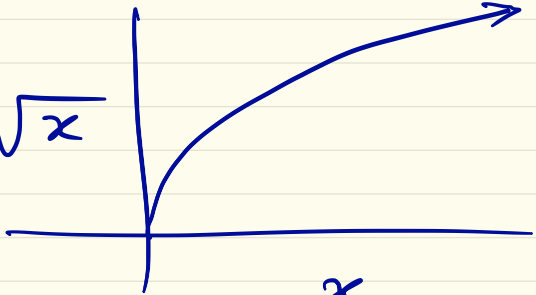
Example: $f: \mathbb{R}_{>0} \longmapsto \mathbb{R}$

- $f(x) = x^\alpha \longleftrightarrow$ convex for $\alpha \geq 1$, $\alpha \leq 0$

concave $0 \leq \alpha \leq 1$



$\sqrt{x}$

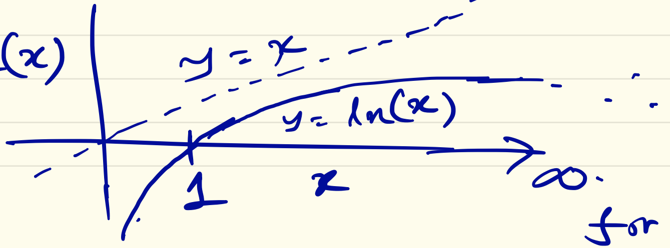


x

- $\log(x)$, $x \in \text{dom}(f) = \mathbb{R}_{>0}$

concave

$\log(x)$



$-\log(x)$

$= \log(1/x)$

for $x>0$ is convex

Example: $f : \mathbb{R}^n \mapsto \mathbb{R}$, is any norm on $\mathbb{R}^n$.

$$\underbrace{\| \theta \underline{x} + (1-\theta)\underline{y} \|}_{f(\theta \underline{x} + (1-\theta)\underline{y})} \leq \underbrace{\| \theta x \| + \| (1-\theta) y \|}_{\substack{\theta \|x\| + (1-\theta)\|y\| \\ \theta f(x) + (1-\theta) f(y)}}$$

Triangle ineq.

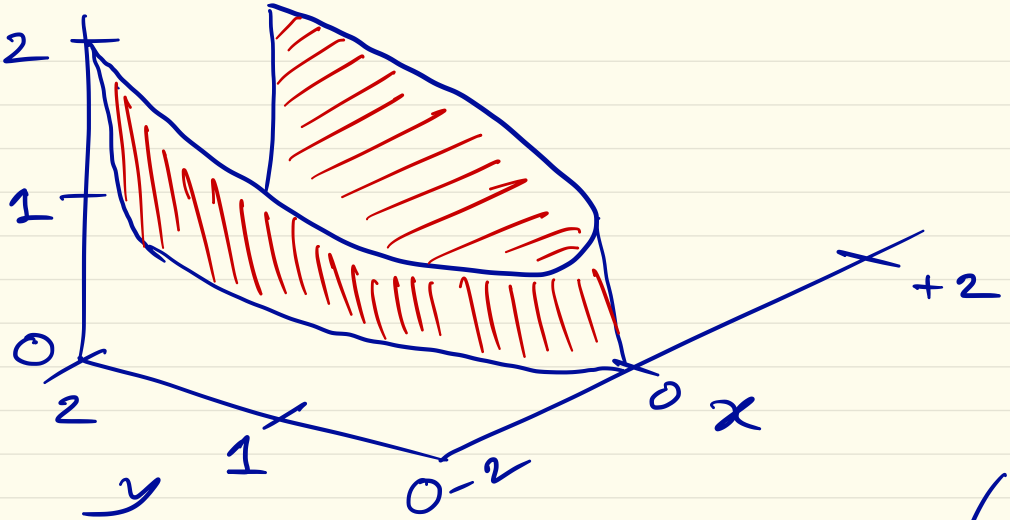
$\therefore$ f is convex

Example: $f : \underbrace{\mathbb{R}}_{x \in} \times \underbrace{\mathbb{R}_{>0}}_{y \in} \mapsto \mathbb{R}$

$$f(x, y) = \text{quadratic-over-linear} = \frac{x^2}{y}$$

$$\nabla^2 f = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \frac{2}{y^3} \begin{bmatrix} y^2 & -xy \\ -xy & x^2 \end{bmatrix}$$

$$= \frac{2}{y^3} \underbrace{\begin{pmatrix} y \\ -x \end{pmatrix}}_{2\times 1} \underbrace{\begin{pmatrix} y \\ -x \end{pmatrix}^T}_{1\times 2} \succcurlyeq 0$$



Example: $f(\underline{x}) = GM(\underline{x}) = \left(\prod_{i=1}^{n} x_i\right)^{1/n}$,

p. 74 in Textbook. $\qquad dom(f) = \mathbb{R}^n_{>0}$

→ concave in $\underline{x} \in dom(f) = \mathbb{R}^n_{>0}$

Sublevel Set / Superlevel set of a f^{n}:

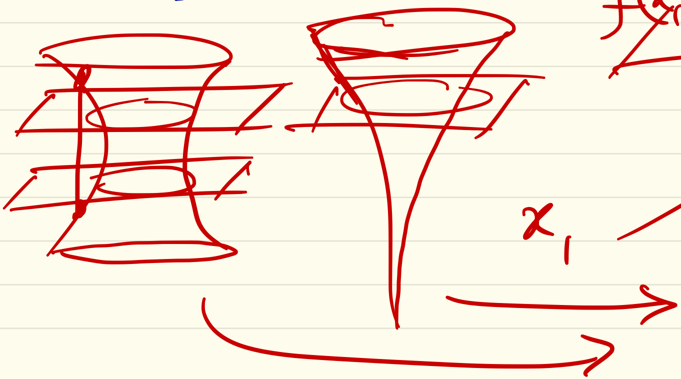
α- sublevel set of $f : \mathbb{R}^n \mapsto \mathbb{R}$ is

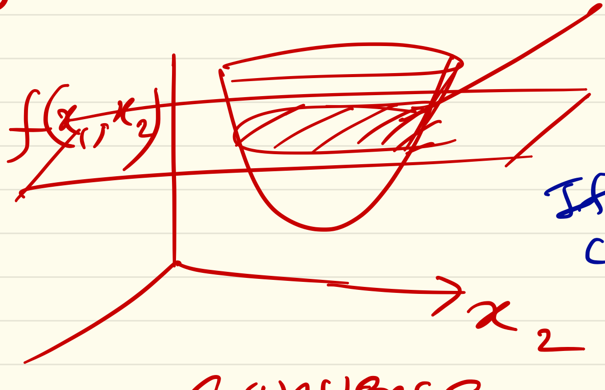
$$C_\alpha := \{ \underline{x} \in \mathrm{dom}(f) \mid f(\underline{x}) \leq \beta \} \leftarrow \text{sublevel set}$$

$$\geq \beta \leftarrow \text{superlevel set.}$$

Claim:

Sublevel set of a convex f^{n} is convex (for any β)





Converse NOT true

If f is concave then α-superlevel sets are convex

Example : Choose $0 \le \alpha \le 1$

Prove that $\mathcal{X} := \{ \underline{x} \in \mathbb{R}^n_{\geq 0} \mid GM(\underline{x}) \geq \alpha AM(\underline{x}) \}$ is convex.

$GM(\underline{x}) - \alpha AM(\underline{x}) \geq \beta$

Proof: Consider the f^{n} :

$$f(\underline{x}) = GM(\underline{x}) - \alpha AM(\underline{x})$$

$$= \underbrace{\left(\prod_{i=1}^{n} x_i \right)^{1/n}}_{\text{concave}} \underbrace{- \alpha \left(\frac{\sum_{i=1}^{n} x_i}{n} \right)}_{\text{concave}}$$

Now,

0- superlevel set of f is concave (That is $\beta = 0$) ($\because$ sum of concave)

$\therefore$ $\mathcal{X}$ is convex.

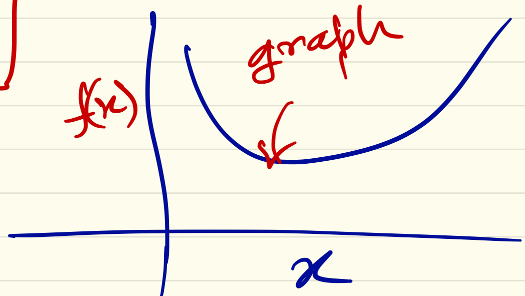
(Just showed set convexity via f^{n} convexity)

Graph: Given $f: \mathbb{R}^n \mapsto \mathbb{R}$

the graph of f is

$$\{(\underline{x}, f(\underline{x})) \mid \underline{x} \in \mathrm{dom}(f)\} \subset \mathbb{R}^{n+1}$$

$\boxed{\text{epi} = \text{above}, \text{ hypo} = \text{below}}$



epigraph of $f: \mathbb{R}^n \mapsto \mathbb{R}$ is

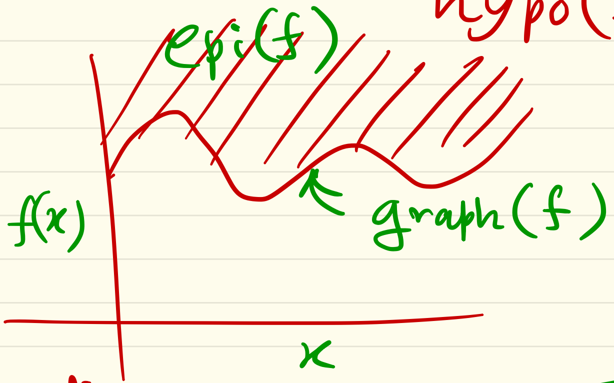
$$\mathrm{epi}(f) := \{(\underline{x}, t) \mid \underline{x} \in \mathrm{dom}(f), f(\underline{x}) \leq t\} \subset \mathbb{R}^{n+1}$$

above the graph

A $f^{\underline{n}}$: f is convex $\iff$ epi(f) is convex

(concave) $\iff$ hypo(f) is "

$$\text{hypo}(f) = \{(x,t) \mid t \leq f(x)\}$$



Example:

$$f: \mathbb{R}^n \times S^n_{++} \mapsto \mathbb{R}$$

$$f(\underline{x}, Y) = \underline{x}^T Y^{-1} \underline{x}$$

Appendix A.5.5 in book

Prove that $f(\underline{x}, Y)$ is convex on dom(f).

One proof:

$$\text{epi}(f) = \{(\underline{x}, Y, t) \mid Y \succ 0,\ \underline{x}^T Y^{-1} \underline{x} \leq t\}$$

$$= \left\{(\underline{x}, Y, t) \;\middle|\; \begin{bmatrix} Y & \underline{x} \\ \underline{x}^T & t \end{bmatrix}_{(n+1)\times(n+1)} \succeq 0,\ Y \succ 0\right\}$$

Schur Complement Lemma:

$$X = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \in S^n, \quad \text{Let } \det(A) \neq 0$$

Then the matrix

$$S = C - B^T A^{-1} B \quad \text{(Schur complement of } A \text{ in } X\text{)}$$

- $X \succ 0 \iff A \succ 0 \ \& \ S \succ 0$
- If $A \succ 0$, then $X \succeq 0 \iff S \succeq 0$

Positive definite

Jensen's inequality:

$$f(\theta \underline{x} + (1-\theta)\underline{y}) \le \theta f(\underline{x}) + (1-\theta) f(\underline{y})$$

$$0 \le \theta \le 1$$

Jensen's original proof:

$$f\left(\frac{\underline{x}+\underline{y}}{2}\right) \le \frac{f(\underline{x})+f(\underline{y})}{2}$$

$$f\left(\sum_{i=1}^{k} \theta_i \underline{x}_i\right) \le \sum_{i=1}^{k} \theta_i f(\underline{x}_i), \quad \sum_{i=1}^{k} \theta_i = 1, \quad \theta_i \geq 0$$

Extension to integral:

$p(\underline{x}) \geq 0 \quad p: \mathbb{R}^n \mapsto \mathbb{R}$ on $S \subseteq \mathrm{dom}(f)$

s.t. $\int p(\underline{x})\, d\underline{x} = 1$

then

$$f\Big(\underbrace{\overbrace{\int_S p(\underline{x})}^{\text{scalar}}\, \underline{x}\, d\underline{x}}_{\text{vector}}\Big) \le \overbrace{\int_S \underbrace{f(\underline{x})}_{\text{scalar}}\, p(x)\, dx}^{\text{scalar}}$$

If $\underline{x} \in \mathrm{dom}(f)$ then

$$\overbrace{f(\underbrace{\mathbb{E}[\underline{x}]}_{\text{vector}})}^{\text{Scalar}} \leq \underbrace{\mathbb{E}[\overbrace{f(\underline{x})}^{\text{scalar}}]}_{\text{scalar}}$$

Original $\text{def}^{\underline{n}}$ of convex $f^{\underline{n}}$ can be thought of $x \in \{x_1, x_2\}$

$$\mathbb{P}(x = x_1) = \theta, \quad \mathbb{P}(x = x_2) = 1 - \theta$$

Appl$^{\underline{n}}$ of Jensen's ~~I~~nequality

$-\log(x)$ is convex for $x > 0$, Take $a, b \geq 0$

$0 \leq \theta \leq 1$

$$\therefore -\log(\theta a + (1-\theta) b) \leq \underbrace{\theta(-\log(a)) + (1-\theta)(-\log(b))}$$

$$= -\log(a^\theta) - \log(b^{1-\theta})$$

$$= -\log(a^\theta b^{1-\theta})$$

$$\Rightarrow \boxed{\theta a + (1-\theta) b \geq a^{\theta} b^{1-\theta}}$$

AM-GM inequality

$$\boxed{\text{For } \theta = 1/2 \rightarrow \quad \frac{a+b}{2} \geq \sqrt{ab}}$$

Take $a = \dfrac{|x_i|^p}{\sum_{j=1}^{n} |x_j|^p}$, $b = \dfrac{|y_i|^q}{\sum_{j=1}^{n} |y_j|^q}$, $\theta = \dfrac{1}{p}$, $p > 1$

$$\frac{1}{p} + \frac{1}{q} = 1$$

$$\underline{x}, \underline{y} \in \mathbb{R}^n$$

$$\theta a + (1-\theta) b \geq a^{\theta} b^{1-\theta}$$

Substitute & sum both sides over index $i = 1, \dots, n$

$$\Rightarrow \sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \left(\sum_{i=1}^{n} |y_i|^q \right)^{1/q}$$

$1/p + 1/q = 1$

This is Hölder's inequality