Lecture #9

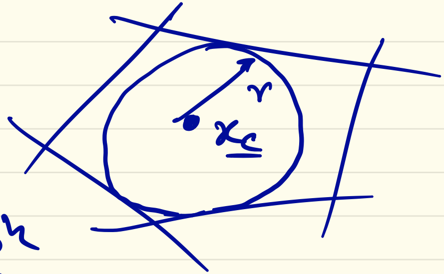
LP

Example: Chebyshev center of polyhedron

Given, $P = \{ \underline{x} \in \mathbb{R}^n \mid a_i^T \underline{x} \leq b_i,\ i = 1, \ldots, m \}$

Find $B = \{ \underline{x}_c + \underline{u} \mid \| \underline{u} \|_2 \leq r \}$

$$\boxed{\begin{aligned} &\max_{(\underline{x}_c, r)} \quad r \\ &\text{s.t.} \quad B \subseteq P \end{aligned}}$$



Optimization variable $= \underline{x}_c \in \mathbb{R}^n$

$r > 0$

Now $\quad a_i^T \underline{x} \leq b_i \quad \forall\, \underline{x} \in \mathcal{B}$ ($\Leftrightarrow$ ball lies within polytope)

$$\Updownarrow$$

$$\sup \{ a_i^T (\underline{x}_c + \underline{u}) \mid \|\underline{u}\| \leq r \}$$

$$= \boxed{\underline{a}_i^T \underline{x}_c + r \|\underline{a}_i\|_2 \leq b_i}$$

(if $> b_i$, then ball spills out)

$$\Updownarrow$$

$$\left.\begin{aligned} &\max_{r>0} \quad r \\ &\text{s.t.} \quad \underline{a}_i^T \underline{x}_c + r\|\underline{a}_i\|_2 \leq b_i, \\ &\qquad\qquad i = 1, \ldots, m \end{aligned}\right\} \underline{\text{LP}}$$

Example:

$$\underbrace{\min_{\underline{x}\in\mathbb{R}^n} \max_{i=1,\dots,m} (\underline{a}_i^T \underline{x} + b_i)}_{\text{piecewise linear}}$$

standard Convex form

$$\min f_0(\underline{x}), \quad \underline{x}\in\mathbb{R}^n$$

$$\text{s.t.} \quad f_i(\underline{x}) \le 0, \quad i=1,\dots,m$$

$$h_j(\underline{x}) = 0, \quad j=1,\dots,p$$

$\Longleftrightarrow$

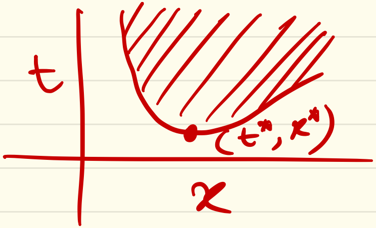
epigraph form

$$\min_{(\underline{x},t)} t$$

$$\text{s.t.} \quad f_0(\underline{x}) - t \le 0$$

$$f_i(\underline{x}) \le 0$$

$$h_j(\underline{x}) = 0$$



$$\min_{(\underline{x},t)} t$$

$$\text{s.t.} \quad \max_{i=1,\dots,m} (\underline{a}_i^T \underline{x} + b_i) \le t$$

$\Updownarrow$

$$\min t$$

$$\text{s.t.} \quad \underline{a}_i^T \underline{x} + b_i \le t, \quad i=1,\dots,m$$

may be analytically solvable

e.g. $\min \; c^T x$ s.t. $\underline{l} \leq \underline{x} \leq \underline{u}$

$\min \; c_i x_i$
s.t. $l_i \leq x_i \leq u_i$

If $c_i > 0$ then $x_i^* = l_i$

If $c_i < 0$, then $x_i^* = u_i$

If $c_i = 0$, then $x_i^* \in [l_i, u_i]$

$\therefore \; (c^T x)_{\text{optimal}} = \underline{l}^T \underline{c}^+ + \underline{u}^T \underline{c}^-$

$c_i^+ := \max\{c_i, 0\}$

$c_i^- := \max\{-c_i, 0\}$

Try yourself:
$\min \; c^T x$
s.t. $\underline{1}^T \underline{x} = 1$
$\underline{x} \geq 0$

Problems involving l_1 & l_∞ norms

l_1 - norm approximation problem:

$$\boxed{\min_{\underline{x} \in \mathbb{R}^n} \|A\underline{x} - \underline{b}\|_1}$$

$$\min_{\binom{x}{y} \in \mathbb{R}^2} \left\| \begin{pmatrix} 2x+3y-1 \\ y \end{pmatrix} \right\|_1 = \underbrace{\min_{\binom{x}{y} \in \mathbb{R}^2} |2x+3y-1| + |y|}$$

$$\Updownarrow$$

Let $t_1 \geq 0$, $t_2 \geq 0$ s.t.

$$\left. \begin{array}{l} |2x+3y-1| \leq t_1 \\ |y| \leq t_2 \end{array} \right\} \Rightarrow \begin{array}{l} \min \ t_1 + t_2 \\ \text{s.t. } |2x+3y-1| \leq t_1 \\ \quad\ \ |y| \leq t_2 \end{array}$$

$$(2x+3y-1) \leq t_1$$

$$-(2x+3y-1) \leq t_1$$

$$y \leq t_2$$

$$-y \leq t_2$$

$$\min_{\underline{x}\in\mathbb{R}^n} \|A\underline{x}-\underline{b}\|_1 \iff \begin{array}{ll}\min & \mathbb{1}^T \underline{t} \\ \text{s.t.} & A\underline{x}-\underline{b} \preceq \underline{t} \\ & -(A\underline{x}-\underline{b}) \preceq \underline{t} \\ & t_1,\dots,t_m \geq 0\end{array}$$

$$\underline{x}\in\mathbb{R}^n$$

$$A\in\mathbb{R}^{m\times n}$$

$$\underline{b}\in\mathbb{R}^m$$

$$\min_{x\in\mathbb{R}^n} \|A\underline{x}-b\|_\infty$$

$$\min_{\binom{x}{y}\in\mathbb{R}^2} \left\|\begin{pmatrix}2x+3y-1\\ y\end{pmatrix}\right\|_\infty$$

$$= \min_{\binom{x}{y}\in\mathbb{R}^2} \max\left(|2x+3y-1|, |y|\right)$$

Trick: Let $t\geq 0$ be s.t.

$$|2x+3y-1|\leq t$$

$$|y|\leq t$$

$$t\geq 0$$

$$\begin{array}{l}\min\ t \\ \text{s.t.} \\ (A\underline{x}-\underline{b})_i \leq t\mathbb{1} \\ -(A\underline{x}-\underline{b})_i \leq t\mathbb{1} \\ t\geq 0\end{array}$$

Try at home.

$$\begin{aligned} \min \ & \|A\underline{x} - \underline{b}\|_1 \\ \text{s.t. } & \|\underline{x}\|_\infty \leq 1 \\ & \underline{x} \in \mathbb{R}^n \end{aligned} \iff \begin{aligned} \min \ & \mathbb{1}^T \underline{y} \\ \text{s.t. } & (-\underline{y}) \preceq A\underline{x} - \underline{b} \preceq +\underline{y} \\ & -\mathbb{1} \preceq \underline{x} \preceq +\mathbb{1} \\ & \underline{y} \in \mathbb{R}^m \end{aligned}$$

Example of QP: (Compute distance betn. 2 polyhedra)

$$\mathcal{P}_1 = \{\underline{x} \in \mathbb{R}^n \mid A_1 \underline{x} \preceq \underline{b}_1\}$$

$$\mathcal{P}_2 = \{\underline{x} \in \mathbb{R}^n \mid A_2 \underline{x} \preceq \underline{b}_2\}$$

$$\min \ \|\underline{x}_1 - \underline{x}_2\|_2^2$$

$$\left.\begin{aligned} \underline{x}_1 \in \mathcal{P}_1 \iff A_1 \underline{x}_1 \preceq \underline{b}_1 \\ \underline{x}_2 \in \mathcal{P}_2 \iff A_2 \underline{x}_2 \preceq \underline{b}_2 \end{aligned}\right\} \ \underline{x}_1, \underline{x}_2 \in \mathbb{R}^n$$

GP (Geometric program) $\underline{x} \in \mathbb{R}^n$

we say $g(\underline{x}) = c\, x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$,

$\text{dom}(g) = \mathbb{R}^n_{>0}$

$c > 0$

$a_i \in \mathbb{R}$

$7 x_1^{3/2} x_2^{-0.9} x_3^{5}$

monomial function of $\underline{x}$

$$f(\underline{x}) = \sum_{k=1}^{K} \underbrace{c_k}_{>0} x_1^{a_{1k}} \cdots x_n^{a_{nk}}$$

is called "posynomial function"

GP (Geometric Program)

$$\min_{\underline{x} \in \mathbb{R}^n_{>0}} f_0(\underline{x})$$

$$\text{s.t.} \quad f_i(\underline{x}) \le 1, \quad i = 1, \dots, m$$

$$h_j(\underline{x}) = 1, \quad j = 1, \dots, p$$

$f_0, \dots, f_m \longrightarrow$ are posynomials

$h_1, \dots, h_p \longrightarrow$ monomials.

e.g.

$$\max \quad \frac{x}{y}$$

$$\text{s.t.} \quad \left.\begin{aligned} &2 \le x \le 3 \\ &x^2 + \frac{3y}{z} \le \sqrt{y} \\ &\frac{x}{y} = z^2 \end{aligned}\right\} \quad x, y, z \geqslant 0$$

LMI or Linear Matrix Inequality in SDP

Standard form of SDP:

$$\min \ \underline{c}^T \underline{x}$$
$$\text{s.t.} \ \boxed{F(\underline{x}) \succeq 0}$$
$$A\underline{x} \leq \underline{b}$$

$$F : \mathbb{R}^n \mapsto S^n_+$$

$$\underline{x} \in \mathbb{R}^n \iff \underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$F(\underline{x}) := F_0 + x_1 F_1 + x_2 F_2 + \dots + x_n F_n \succeq 0$$

where $F_0, F_1, \dots, F_n \in S_n$

Example:

$$\boxed{1 - x^2 \geq 0}$$

$$\Updownarrow$$

$$\begin{bmatrix} 1 & x \\ x & 1 \end{bmatrix} \succeq 0$$

$$= \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{F_0} + x \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{F_1}$$

Standard form

$$F(x) \succ 0$$

Decomposition

$$F(\underline{x}) = F_0 + x_1 F_1 + \dots + x_n F_n$$

1 LMI $\iff$ polynomial inequalities corresponding all principal minors ≥ 0 (not just leading)

Pos. def. in different dimensions

1 D

$x_1 \geq 0$

$x_1 \geq 0$

2 D

$$\begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix} \geq 0$$

$$\Updownarrow$$

$$\begin{cases} x_1 \geq 0, \; x_3 \geq 0 \\ x_1 x_3 - x_2^2 \geq 0 \end{cases}$$

(Notice that x_2 can be negative)

3 D

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_4 & x_5 \\ x_3 & x_5 & x_6 \end{bmatrix} \geq 0$$

$$\Updownarrow$$

$$\begin{cases} x_1 \geq 0, \; x_4 \geq 0, \; x_6 \geq 0 \\ x_1 x_4 - x_2^2 \geq 0 \\ x_4 x_6 - x_5^2 \geq 0 \\ x_1 x_6 - x_3^2 \geq 0 \\ x_1 x_4 x_6 + 2 x_2 x_3 x_5 - x_1 x_5^2 - x_6 x_2^2 - x_4 x_3^2 \geq 0 \end{cases}$$

(Notice that x_2, x_3, x_5 can be negative)

$F \geq 0$

$$\Updownarrow$$

$F_{11} \geq 0$

$$\begin{vmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{vmatrix} > 0 \quad \ldots$$

Example:

$$\left.\begin{array}{l} y>0 \\ y-x^2>0 \end{array}\right\} \Longleftrightarrow \underbrace{\begin{bmatrix} y & x \\ x & 1 \end{bmatrix} > 0}_{\text{general form}}$$

Standard/decompos^n form:

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + y\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + x\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} > 0$$

Example

$$x_1^2 + x_2^2 < 1 \Longleftrightarrow \begin{bmatrix} 1 & 0 & x_1 \\ 0 & 1 & x_2 \\ x_1 & x_2 & 1 \end{bmatrix} > 0$$

Leading minors:

$$1 > 0$$

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} > 0$$

$$1\begin{vmatrix} 1 & x_2 \\ x_2 & 1 \end{vmatrix} - 0\begin{vmatrix} 1 & x_1 \\ x_1 & 1 \end{vmatrix} + x_1\begin{vmatrix} 0 & 1 \\ x_1 & x_2 \end{vmatrix} > 0$$

Example: minimize max. eig.

Let $A_0, A_1, \ldots, A_n \in S^n$

Let $A(\underline{x}) = A_0 + A_1 x_1 + \ldots + A_n x_n$

clearly $A(\underline{x}) \in S^n$

$$\boxed{\min_{\underline{x} \in \mathbb{R}^n} \lambda_{max}(A(\underline{x}))}$$

Recall (linear algebra) if $M \in S^n$

then $\lambda_{max}(M) \leq t$

$$\Updownarrow$$

$$M - tI \preceq 0$$

$$\left.\begin{aligned} &\min_{\underline{x} \in \mathbb{R}^n, t} t \\ &\text{s.t. } \boxed{A(\underline{x}) - tI \preceq 0} \end{aligned}\right\} \text{This is an SDP}$$

LMI constraint